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THE UNIVERSITY OF ALBERTA

QUEUING THEORY

AND

ACTUAL TELEPHONE EXCHANGE EXPERIENCE

by



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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled QUEUING THEORY AND ACTUAL TELEPHONE EXCHANGE EXPERIENCE submitted by SANDRA FOSTER SCHELLENBERG in partial fulfilment of the requirements for the degree of Master of Science.

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ABSTRACT

This thesis consists of two main parts. The first is an organized presentation of some results in queuing theory that are applicable in the analysis of telephone trunking problems. Here four situations, all assuming Poisson input and exponential holding times, are discussed. These are the case of the first-come-first-served single channel queue, the case of an infinite number of servers, the case of a finite number of servers where one waiting line is formed on a first-come-first-served basis and finally the case of a finite number of servers where an arriving item is lost if all servers are busy.

The second part of the thesis is an attempt to analyse some actual data obtained from Alberta Government Telephones on the calls terminating at the 699 77— Sherwood Park telephones from 9 A.M. to 5 P.M. over about an eight day period. Here the interarrival times and busy times are calculated and the results compared with the exponential assumptions made previously.

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CHAPTER I

INTRODUCTION

1.1 A Discussion of Queuing Theory

When people arrive at a facility that provides some necessary or desirable service, whether it be a check-out counter in a grocery store or a subway station, they frequently encounter a queue or waiting line. The study of these waiting lines, queuing theory, is a mathematical descriptive theory that attempts to formulate, interpret and predict queuing behavior for the purpose of better understanding this bothersome, but seemingly inevitable, aspect of our daily lives and for the sake of introducing remedies.

The importance of studying and developing such a theory is clearly evident in the amount of time an individual is made to wait in his daily activities. If one is not properly occupied many queuing situations amount to an appreciable depreciation of the enjoyment of life. But by anticipation and good planning the agony and waste of queues could be minimized.

Queues, however, are an unavoidable part of our modern society. With an ever increasing population, the number of people demanding the same services will continue to grow and unless these services are increased, which in some cases can only be done at enormous costs, the amount of time an individual spends in a queue in his lifetime will increase considerably. Thus it becomes evermore necessary to understand and anticipate queues and their subtleties.

However, in attempting to do this it must be kept in mind that most queuing systems involve human beings. Studying causes and remedies of queuing problems cannot be completely divorced from consideration of human factors and their influence on the problem. Nor can remedies be applied without regard to the fact that it is the people using the facilities who matter in the final analysis.

1.2 The Application of Queuing Theory to Telephony

It is interesting and perhaps of particular importance that the origin of queuing theory is to be found in telephone-network congestion problems. Some basic ideas of telephony that are of particular interest to the remainder of this presentation are described as follows.

Let us assume that we have a set of devices (telephones, trunks, lines, etc.) that are all accessible to a group of sources (telephone subscribers). When a caller from a certain group wishes to make a connection, he may do so through any one of the free lines. If all lines are engaged then the caller either waits or his call is lost; that is he receives a busy signal. In either case, the probability of a call waiting a certain time or getting lost is of interest.

Much of the early work on telephone theory was done by A.K. Erlang.¹ If there is a limited source of N subscribers of whom

¹Brockmeyer, E., H.L. Halstrom and A. Jensen: "The Life and Works of A.K. Erlang" (all of Erlang's papers translated into English), Copenhagen Telephone Company, Copenhagen, 1948.

n are occupied making calls, Erlang assumed that the probability of a new call being initiated in the time interval $(t, t+dt)$ is $(N-n)\lambda dt$ where λ is the call rate per unit time of a caller. If no consideration is given to the number of occupied independent callers, then the probability of a new independent call being initiated is λdt , leading to the exponential distribution $\lambda e^{-\lambda t}$ for the distribution of intervals between successive calls. This obviously gives a Poisson distribution $(\lambda t)^n e^{-\lambda t} / n!$ for the probability of exactly n calls arriving during time t with a mean of λt calls. Similarly, if $1/\mu$ is the average duration of a call, the probability of a call terminating during $(t, t+dt)$ is μdt which is independent of the length of the call and of other calls. This in turn implies that call durations are distributed according to $\mu e^{-\mu t}$.

Erlang developed telephone traffic theory, using these assumptions, and obtained results for the probability of different numbers of calls waiting and for the waiting time when the system is in equilibrium and the probability of loss for the loss system. He assumed Poisson inputs from unlimited sources and either exponential or constant holding times.

1.3 The Relationship to Markov Processes

The main assumption in the above discussion is that the interarrival times and service times are exponential. This simplifies the analysis of the situation as then the process $\{X(t)\}$, where $X(t)$ is the number of calls in the particular system we are studying at time t , is certainly Markovian.

Even if only one of the interarrival times or the holding times is exponential we may still use the Markov approach.

If the interarrival times are exponential, the $\{X(t)\}$ process is considered just when a customer finishes service. In this way we generate a sequence of integer values

$$X(T_1), X(T_2), X(T_3), \dots$$

where $T_1, T_2, T_3, \dots$ are the successive times of completion of service. The sequence $\{X(T_n)\}$ forms a discrete time process that (because of the Poisson nature of the input) describes a Markov chain.

Similarly, the exponential nature of the service times will induce a Markov chain at the times of arrival of new customers.

However, if neither the interarrival times nor the service times are exponential, we do not have an embedded Markov chain and the situation is much more difficult. Very little has been written about this case, except perhaps for one server systems. Some examples of contributions to this problem can be found in D.V. Lindley's paper, "The Theory of Queues with a Single Server", and in papers by S.O. Rice and V.E. Benes that appear in the Bell System Technical Journals.

1.4 Outline of the Thesis

The ensuing material will be divided into two parts. In the first part there will be a development of some of the mathematical ideas that have been established in telephone theory, beginning with a

brief discussion of Erlang's original model.

Here he describes the situation of a first-come-first-served single channel queue with a Poisson input with parameter λ and exponential holding times with parameter μ . By solving a set of differential equations that give the change in the probability of n calls in the system at any time t with respect to time we will obtain an expression for this probability. By setting the time derivatives equal to zero and eliminating time from these differential equations it will be possible to describe the situation where the system eventually reaches an equilibrium state independent of time, giving the probability that n calls are in the system in this steady state. Then we will approach the question of waiting times in order to obtain the probability of an arriving call being delayed longer than time t . In this way it could perhaps be determined whether it would be profitable for a customer to wait or not.

Following this discussion, the case of an infinite number of channels that are all accessible to a certain group of subscribers, and thus a situation which involves no loss or waiting, will be considered. Again the probability of n calls in the system at any time t and in the equilibrium state will be determined.

Next we will consider the situation of a finite number of channels that are all accessible to a certain group of subscribers where callers will form a single first-come-first-served waiting line if all channels are occupied. In this case, it is quite difficult to determine the probability of n items in the system at time t from the

differential equations. However, once this has been accomplished, a straightforward discussion, much similar to that of Erlang's model, will be given to determine the equilibrium probabilities and the waiting times both in the queue and in the system.

Finally the situation of a finite number of channels that are all accessible to a certain group of subscribers where each caller will hang up if all channels are occupied will be considered. Here the differential equations describing the change in the probability of n calls in the system at any time t with respect to time will not be solved as they involve only a slight modification in those of the previous case. The steady-state probabilities will be given however, and also the probability of a call being lost to the system (Erlang's loss formula) will be obtained.

In the second part of this presentation an attempt will be made to analyse some actual data obtained from Alberta Government Telephones on the calls terminating at the 699 77— Sherwood Park telephones from 9 A.M. to 5 P.M. over about an eight day period. Here we will begin with an explanation of the data, which is in the form of a pen recorder chart, and then outline the steps that were followed in interpreting this data, in calculating the interarrival times and busy times and in estimating the necessary parameters. Finally, the results obtained will be discussed and conclusions drawn as to how well these results compare with the exponential assumptions made previously.

CHAPTER II

SOME RESULTS IN QUEUING THEORY

2.1 A Discussion of Erlang's Original Model

Turning now to the discussion of Erlang's original model, let $P_n(t)$ be the probability of n calls in the system at time t ; that is one customer being served and $(n-1)$ waiting. We immediately have the equations

$$\begin{aligned} P_n(t+\Delta t) = & P_n(t) [(1-\lambda\Delta t)(1-\mu\Delta t)] \\ & + P_{n-1}(t)\lambda\Delta t(1-\mu\Delta t) \\ & + P_{n+1}(t)\mu\Delta t(1-\lambda\Delta t) + o(\Delta t) \quad \text{for } n \geq 1, \end{aligned} \quad (1)$$

$$P_0(t+\Delta t) = P_0(t)(1-\lambda\Delta t) + P_1(t)\mu\Delta t(1-\lambda\Delta t) + o(\Delta t).$$

We arrive at these equations by observing that the probability of n calls in the system at time $t + \Delta t$ is equal to the probability of n calls at time t and no arrivals or departures during Δt or the probability of $(n - 1)$ calls at time t and one arrival but no departures during Δt or the probability of $(n + 1)$ calls at time t and one departure but no arrivals during Δt or the probability of more than one change during Δt , which is $o(\Delta t)$.

By transposing and passing to the limit with respect to Δt we obtain the system:

$$\frac{dP_n(t)}{dt} = -(\lambda + \mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t)$$

for $n \geq 1$, (2)

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) ,$$

which can be solved to obtain an expression for $P_n(t)$.

If we are able to determine the generating function

$$P(z, t) = \sum_{n=0}^{\infty} P_n(t) z^n, \quad |z| \leq 1 , \quad (3)$$

the probabilities $P_n(t)$ which we seek are the coefficients of z^n for every n . However, instead of determining $P(z, t)$ we will obtain an expression for its Laplace transform $P^*(z, s)$ and calculate its coefficients, observing that $P_n^*(s)$ is the transform of $P_n(t)$.

We begin by multiplying each equation of (2) by z^{n+1} and summing over n , obtaining

$$z \frac{\partial P(z, t)}{\partial t} = (1-z) [(\mu - \lambda z)P(z, t) - \mu P_0(t)] . \quad (4)$$

If we assume $P_i(0) = 1$, then $P(z, 0) = z^i$.

Taking the Laplace transform of this expression and solving for $P^*(z, s)$ we obtain

$$P^*(z,s) = \frac{z^{i+1} - \mu(1-z)P_O^*(s)}{sz - (1-z)(\mu - \lambda z)} . \quad (5)$$

Since $P^*(z,s)$ converges inside and on the unit circle for $\text{Re}(s) > 0$, the zeros of the numerator must correspond with those of the denominator in this domain.

The zeros of the denominator are

$$\alpha_k(s) = \frac{\lambda + \mu + s \pm [(\lambda + \mu + s)^2 - 4\lambda\mu]^{\frac{1}{2}}}{2\lambda} , \quad k = 1, 2 , \quad (6)$$

where $\alpha_1(s)$ has the positive sign before the radical.

In order to determine whether these zeros are inside the unit circle we will apply Rouché's Theorem to the denominator $sz - (1-z)(\mu - \lambda z)$ letting $f(z) = (\lambda + \mu + s)z$ and $g(z) = \lambda z^2 + \mu$. Clearly for $|z| = 1$, $|f(z)| > |g(z)|$ and thus $sz - (1-z)(\mu - \lambda z)$ has the same number of zeros inside the unit circle as $f(z)$, namely one. This zero must be $\alpha_2(s)$ since $|\alpha_2(s)| < |\alpha_1(s)|$ and there is no zero on $|z| = 1$.

Thus the numerator $z^{i+1} - \mu(1-z)P_O^*(s)$ must vanish for $z = \alpha_2$, that is

$$\alpha_2^{i+1} - \mu(1-\alpha_2)P_O^*(s) = 0 ,$$

implying

$$P_O^*(s) = \frac{\alpha_2^{i+1}}{\mu(1-\alpha_2)} . \quad (7)$$

Substituting this into the expression for $P^*(z,s)$ gives

$$P^*(z,s) = \frac{z^{i+1} - [(1-z)\alpha_2^{i+1}/(1-\alpha_2)]}{-\lambda(z-\alpha_1)(z-\alpha_2)} \quad (8)$$

$$\text{as } \alpha_1 + \alpha_2 = \frac{\lambda + \mu + s}{\lambda}, \quad \alpha_1 \alpha_2 = \mu/\lambda \quad \text{and} \quad s = -\lambda(1-\alpha_1)(1-\alpha_2) .$$

When the numerator is multiplied through by $1 - \alpha_2$, it simplifies and factors giving

$$\begin{aligned} P^*(z,s) &= \frac{(z-\alpha_2)(z^{i+\alpha_2} z^{i-1} + \dots + \alpha_2^i) - z\alpha_2(z-\alpha_2)(z^{i-1+\alpha_2} z^{i-2} + \dots + \alpha_2^{i-1})}{\lambda\alpha_1(z-\alpha_2)(1-z/\alpha_1)(1-\alpha_2)} \\ &= \frac{(z^{i+\alpha_2} z^{i-1} + \dots + \alpha_2^i) - z\alpha_2(z^{i-1+\alpha_2} z^{i-2} + \dots + \alpha_2^{i-1})}{\lambda\alpha_1(1-z/\alpha_1)(1-\alpha_2)} \\ &= \frac{(1-\alpha_2)(z^{i+\alpha_2} z^{i-1} + \dots + \alpha_2^i) + \alpha_2^{i+1}}{\lambda\alpha_1(1-z/\alpha_1)(1-\alpha_2)} \\ &= \frac{1}{\lambda\alpha_1} (z^{i+\alpha_2} z^{i-1} + \dots + \alpha_2^i) \sum_{k=0}^{\infty} (z/\alpha_1)^k \\ &\quad + \frac{\alpha_2^{i+1}}{\lambda\alpha_1(1-\alpha_2)} \sum_{k=0}^{\infty} (z/\alpha_1)^k \end{aligned} \quad (9)$$

$$\text{where } \sum_{k=0}^{\infty} (z/\alpha_1)^k = \frac{1}{1-z/\alpha_1} \quad \text{as } |z/\alpha_1| < 1 .$$

The coefficient of z^n in the first term of the above statement is

$$\frac{1}{\lambda \alpha_1^{n-i+1}} + \frac{\mu/\lambda}{\lambda \alpha_1^{n-i+3}} + \frac{(\mu/\lambda)^2}{\lambda \alpha_1^{n-i+5}} + \dots + \frac{(\mu/\lambda)^i}{\lambda \alpha_1^{n+i+1}}, \quad n \geq i, \quad (10)$$

$$\frac{(\mu/\lambda)^{i-n}}{\lambda \alpha_1^{i-n+1}} + \frac{(\mu/\lambda)^{i-(n-1)}}{\lambda \alpha_1^{i-n+3}} + \dots + \frac{(\mu/\lambda)^i}{\lambda \alpha_1^{i+n+1}}, \quad n < i, \quad (11)$$

and the coefficient of z^n in the second term of the above statement is

$$\begin{aligned} \frac{\alpha_2^{i+1}}{\lambda \alpha_1^{n+1} (1-\alpha_2)} &= \frac{\alpha_2^{i+1}}{\lambda \alpha_1^{n+1}} (1 + \alpha_2 + \alpha_2^2 + \dots), \text{ as } |\alpha_2| < 1, \\ &= \frac{1}{\lambda} (\lambda/\mu)^{n+1} \sum_{k=n+i+2}^{\infty} (\mu/\lambda)^k \frac{1}{\alpha_1^k}. \end{aligned} \quad (12)$$

Thus for $n \geq i$,

$$P_n^*(s) = 1/\lambda \left[\frac{1}{\alpha_1^{n-i+1}} + \frac{\mu/\lambda}{\alpha_1^{n-i+3}} + \dots + \frac{(\mu/\lambda)^i}{\alpha_1^{n+i+1}} + (\lambda/\mu)^{n+1} \sum_{k=n+i+2}^{\infty} (\mu/\lambda)^k \frac{1}{\alpha_1^k} \right] \quad (13)$$

and thus we will have $P_n(t)$ by taking the inverse transform of $P_n^*(s)$, that is

$$P_n(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{-st} P_n^*(s) ds. \quad (14)$$

To determine this we observe that $[\frac{s+\sqrt{s^2-4\lambda\mu}}{2\lambda}]^{-\nu}$ is the Laplace transform of²

$$\begin{aligned} & (2\lambda)^\nu (2\sqrt{\lambda\mu})^{-\nu} t^{-1} I_\nu(2\sqrt{\lambda\mu} t) \\ &= \nu (\sqrt{\lambda/\mu})^\nu t^{-1} I_\nu(2\sqrt{\lambda\mu} t) \end{aligned} \quad (15)$$

where $I_\nu(z)$ is the modified Bessel function of the first kind given by

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{k! \Gamma(\nu+k+1)}. \quad (16)$$

Thus as $\alpha_1(s) = \frac{(\lambda+\mu+s)+[(\lambda+\mu+s)^2-4\lambda\mu]^{\frac{1}{2}}}{2\lambda}$ and as $f^*(s)$ is the Laplace transform of $f(t)$ implies that $f^*(s+a)$ is the Laplace transform of $e^{-at}f(t)$ we have

$$\begin{aligned} P_n(t) &= \frac{e^{-(\lambda+\mu)t}}{\lambda} [(\sqrt{\lambda/\mu})^{n-i+1} (n-i+1)t^{-1} I_{n-i+1}(2\sqrt{\lambda\mu} t) \\ &\quad + \mu/\lambda (\sqrt{\lambda/\mu})^{n-i+3} (n-i+3)t^{-1} I_{n-i+3}(2\sqrt{\lambda\mu} t)] \end{aligned}$$

²"Bateman Manuscript Project", California Institute of Technology, McGraw Hill Book Co., Inc., New York, 1954, Vol. I, 4.16, (4).

$$\begin{aligned}
& + \dots + (\mu/\lambda)^i (\sqrt{\lambda/\mu})^{n+i+1} (n+i+1)t^{-1} I_{n+i+1}(2\sqrt{\lambda\mu} t) \\
& + (\lambda/\mu)^{n+1} \sum_{k=n+i+2}^{\infty} (\sqrt{\mu/\lambda})^k k t^{-1} I_k(2\sqrt{\lambda\mu} t)] . \quad (17)
\end{aligned}$$

Substituting the well-known relation $\frac{2v}{z} I_v(z) = I_{v-1}(z) - I_{v+1}(z)$ one has

$$\begin{aligned}
P_n(t) &= \frac{e^{-(\lambda+\mu)t}}{\lambda} \{ (\sqrt{\lambda/\mu})^{n-i+1} \sqrt{\lambda\mu} [I_{n-i}(2\sqrt{\lambda\mu} t) - I_{n-i+2}(2\sqrt{\lambda\mu} t)] \\
&+ (\sqrt{\lambda/\mu})^{n-i+1} \sqrt{\lambda\mu} [I_{n-i+2}(2\sqrt{\lambda\mu} t) - I_{n-i+4}(2\sqrt{\lambda\mu} t)] \\
&+ \dots \\
&+ (\sqrt{\lambda/\mu})^{n-i+1} \sqrt{\lambda\mu} [I_{n+i}(2\sqrt{\lambda\mu} t) - I_{n+i+2}(2\sqrt{\lambda\mu} t)] \\
&+ (\lambda/\mu)^{n+1} \sum_{k=n+i+2}^{\infty} (\sqrt{\mu/\lambda})^k \sqrt{\lambda\mu} [I_{k-1}(2\sqrt{\lambda\mu} t) - I_{k+1}(2\sqrt{\lambda\mu} t)] \} . \quad (18)
\end{aligned}$$

But

$$\begin{aligned}
& (\lambda/\mu)^{n+1} \sum_{k=n+i+2}^{\infty} (\sqrt{\mu/\lambda})^k [I_{k-1}(2\sqrt{\lambda\mu} t) - I_{k+1}(2\sqrt{\lambda\mu} t)] \\
&= (\lambda/\mu)^{n+1} [(\sqrt{\mu/\lambda})^{n+i+2} I_{n+i+1}(2\sqrt{\lambda\mu} t) + \sqrt{\mu/\lambda} \sum_{k=n+i+2}^{\infty} (\sqrt{\mu/\lambda})^k I_k(2\sqrt{\lambda\mu} t) \\
&+ (\sqrt{\mu/\lambda})^{n+i+1} I_{n+i+2}(2\sqrt{\lambda\mu} t) - \sqrt{\lambda/\mu} \sum_{k=n+i+2}^{\infty} (\sqrt{\mu/\lambda})^k I_k(2\sqrt{\lambda\mu} t)
\end{aligned}$$

$$\begin{aligned}
&= (\sqrt{\lambda/\mu})^{n-i} I_{n+i+1}(2\sqrt{\lambda\mu} t) + (\sqrt{\lambda/\mu})^{n-i+1} I_{n+i+2}(2\sqrt{\lambda\mu} t) \\
&\quad + (1-\lambda/\mu) (\lambda/\mu)^n \sqrt{\lambda/\mu} \sum_{k=n+i+2}^{\infty} (\sqrt{\mu/\lambda})^k I_k(2\sqrt{\lambda\mu} t) . \tag{19}
\end{aligned}$$

Thus, for $n \geq i$,

$$\begin{aligned}
P_n(t) &= e^{-(\lambda+\mu)t} [(\sqrt{\mu/\lambda})^{n-i} I_{n-i}(2\sqrt{\lambda\mu} t) \\
&\quad + (\sqrt{\mu/\lambda})^{i-n+1} I_{n+i+1}(2\sqrt{\lambda\mu} t) + (1-\lambda/\mu) (\lambda/\mu)^n \sum_{k=n+i+2}^{\infty} (\sqrt{\mu/\lambda})^k I_k(2\sqrt{\lambda\mu} t)] . \tag{20}
\end{aligned}$$

When $n < i$ we have

$$\begin{aligned}
P_n^*(s) &= 1/\lambda \left[\frac{(\mu/\lambda)^{i-n}}{\alpha_1^{i-n+1}} + \frac{(\mu/\lambda)^{i-(n-1)}}{\alpha_1^{i-n+3}} \right. \\
&\quad \left. + \dots + \frac{(\mu/\lambda)^i}{\alpha_1^{i+n+1}} + (\lambda/\mu)^{n+1} \sum_{k=n+i+2}^{\infty} (\mu/\lambda)^k \frac{1}{\alpha_1^k} \right] .
\end{aligned}$$

Thus

$$\begin{aligned}
P_n(t) &= \frac{e^{-(\lambda+\mu)t}}{\lambda} [(\mu/\lambda)^{i-n} (\sqrt{\lambda/\mu})^{i-n+1} (i-n+1)t^{-1} I_{i-n+1}(2\sqrt{\lambda\mu} t) \\
&\quad + \dots + (\mu/\lambda)^i (\sqrt{\lambda/\mu})^{i+n+1} (i+n+1)t^{-1} I_{i+n+1}(2\sqrt{\lambda\mu} t) \\
&\quad + (\lambda/\mu)^{n+1} \sum_{k=n+i+2}^{\infty} (\sqrt{\mu/\lambda})^k k t^{-1} I_k(2\sqrt{\lambda\mu} t)]
\end{aligned}$$

$$\begin{aligned}
&= \frac{e^{-(\lambda+\mu)t}}{\lambda} \{ (\sqrt{\lambda/\mu})^{n-i+1} \sqrt{\lambda\mu} [I_{i-n}(2\sqrt{\lambda\mu}t) - I_{i-n+2}(2\sqrt{\lambda\mu}t)] \\
&\quad + \dots + (\sqrt{\lambda/\mu})^{n-i+1} \sqrt{\lambda\mu} [I_{n+i}(2\sqrt{\lambda\mu}t) - I_{n+i+2}(2\sqrt{\lambda\mu}t)] \\
&\quad + (\lambda/\mu)^{n+1} \sum_{k=n+i+2}^{\infty} (\sqrt{\mu/\lambda})^k \sqrt{\lambda\mu} [I_{k-1}(2\sqrt{\lambda\mu}t) - I_{k+1}(2\sqrt{\lambda\mu}t)] \}
\end{aligned}
\tag{21}$$

which agrees with the result for $n \geq i$ as $I_{-n} = I_n$.

For many problems time dependence is not the typical situation to be studied. After a long period of operation the system acquires a pattern of behaviour which, although describable in terms of probabilities, does not depend on time. Thus as $t \rightarrow \infty$ this situation may be expected to exist for many systems. The question of whether such $p_n = \lim_{t \rightarrow \infty} P_n(t)$ exist is a main concern of ergodic analysis where one actually looks for the probabilities p_n which describe the steady or equilibrium state. But one also notes that the requirement that the probabilities $P_n(t)$ do not change with time is the definition of the steady state.

The changing of $P_n(t)$ with respect to t is described by the derivative $P'_n(t)$, and the steady state condition requires that $P'_n(t) = 0$. Thus essentially we have two ways of obtaining the steady-state probabilities:

1. from $P'_n(t) = 0$ and
2. $\lim_{t \rightarrow \infty} P_n(t)$.

By setting the time derivatives equal to zero and eliminating time from the system of equations (2) we obtain after transposing

$$(\lambda + \mu)p_n = \lambda p_{n-1} + \mu p_{n+1}, \quad n \geq 1,$$

$$\lambda p_0 = \mu p_1. \quad (22)$$

From here we obtain

$$p_n = \rho^n p_0, \quad \text{where } \rho = \lambda/\mu. \quad (23)$$

Since $\sum_{n=0}^{\infty} p_n = 1$,

$$\sum_{n=0}^{\infty} \rho^n p_0 = 1$$

$$\Rightarrow \sum_{n=0}^{\infty} p_0 \rho^n = p_0 \sum_{n=0}^{\infty} \rho^n = \frac{p_0}{1-\rho} = 1 \quad \text{if } \rho = \lambda/\mu < 1$$

$$\Rightarrow p_n = \rho^n (1-\rho) \quad \text{if } \rho = \lambda/\mu < 1. \quad (24)$$

Thus in the case where the traffic intensity factor $\rho = \lambda/\mu < 1$ we have the steady state probabilities satisfying a geometric distribution. In the case where $\lambda/\mu \geq 1$ the number waiting in the system would become infinite with time and no steady state occurs.

Continuing in the case $\lambda/\mu < 1$, the expected number in the system is given by

$$\begin{aligned}
L &= \sum_{n=0}^{\infty} n p_n = (1-\rho) \sum_{n=0}^{\infty} n \rho^n \\
&= (1-\rho) \rho \frac{d}{d\rho} \sum_{n=0}^{\infty} \rho^n \\
&= \frac{\rho}{1-\rho} .
\end{aligned} \tag{25}$$

Fluctuations in the number waiting can in fact occur in the steady-state and this can best be seen by calculating the variance

$$\begin{aligned}
\sum_{n=0}^{\infty} (n-L)^2 p_n &= \sum_{n=0}^{\infty} n^2 p_n - \left(\frac{\rho}{1-\rho}\right)^2 \\
&= (1-\rho) \sum_{n=0}^{\infty} n^2 \rho^n - \left(\frac{\rho}{1-\rho}\right)^2 \\
&= (1-\rho) \rho \frac{d}{d\rho} \rho \frac{d}{d\rho} \sum_{n=0}^{\infty} \rho^n - \left(\frac{\rho}{1-\rho}\right)^2 \\
&= \frac{\rho}{1-\rho} + \frac{2\rho^2}{(1-\rho)^2} - \left(\frac{\rho}{1-\rho}\right)^2 \\
&= \frac{\rho}{1-\rho} + \left(\frac{\rho}{1-\rho}\right)^2 \\
&= L + L^2 .
\end{aligned} \tag{26}$$

The expected number in the line is given by

$$L_q = \sum_{n=1}^{\infty} (n-1) p_n = L - \rho = \frac{\rho}{1-\rho} - \rho = \frac{\rho^2}{1-\rho} . \tag{27}$$

The logical question to ask at this point is how long will the individuals in the queue have to wait? To answer this question suppose that an arriving customer finds n customers ahead of him. His waiting time will clearly be a random variable that is the sum of the service-time random variables of the n individuals ahead. We know that all the service-time distributions are identical exponential distributions with parameter μ and that because of the "forgetfulness" property of the exponential distribution, the remaining service time of the customer in service satisfies the same law as the service time of those who have not yet been served.

If we consider T_i as the service time of the i th customer then

$$P\{(n+1)\text{st customer waits for a time } \leq t\}$$

$$= W_n(t) = P\{T_1 + T_2 + \dots + T_n \leq t\} .$$

The characteristic function of the exponential distribution is $(\frac{\mu}{\mu - i\theta})$ and thus the characteristic function of the distribution of the waiting time of the $(n+1)$ st customer is $(\frac{\mu}{\mu - i\theta})^n$, thus giving a density of

$$\frac{\mu(\mu t)^{n-1} e^{-\mu t}}{(n-1)!}$$

which implies

$$W_n(t) = \int_0^t \frac{\mu(\mu x)^{n-1} e^{-\mu x}}{(n-1)!} dx . \quad (28)$$

The density function of the distribution of the waiting time T of any customer in the line is given by

$$\begin{aligned} w(t) &= \sum_{n=1}^{\infty} \rho^n (1-\rho) \frac{\mu(\mu t)^{n-1} e^{-\mu t}}{(n-1)!} \\ &= \mu \rho (1-\rho) e^{-(1-\rho)\mu t} \end{aligned} \quad (29)$$

and thus we can calculate the average waiting time as

$$\begin{aligned} &\int_0^{\infty} \mu \rho (1-\rho) e^{-(1-\rho)\mu t} t dt \\ &= \mu \rho (1-\rho) \int_0^{\infty} t e^{-(1-\rho)\mu t} dt \\ &= \frac{\rho}{\mu(1-\rho)} . \end{aligned} \quad (30)$$

Also, the probability of waiting no more than time t can be obtained as

$$\begin{aligned} P(T \leq t) &= 1 - P(T > t) = 1 - \int_t^{\infty} \mu \rho (1-\rho) e^{-(1-\rho)\mu x} dx \\ &= 1 - \rho e^{-(1-\rho)\mu t} . \end{aligned} \quad (31)$$

If instead of the waiting time in the line we are interested in

the total length of time spent in the system we can obtain in an analogous manner the density function of the distribution of the total time in the system of the $(n+1)$ st customer as

$$\frac{\mu(\mu t)^n e^{-\mu t}}{n!} \quad (32)$$

which implies that the density function of the distribution of the total time in the system of any customer is

$$\mu(1-\rho)e^{-(1-\rho)\mu t} . \quad (33)$$

Thus the probability of being in the system no more than time t is

$$1 - e^{-(1-\rho)\mu t} \quad (34)$$

and the expected time in the system is

$$\int_0^{\infty} \mu(1-\rho)e^{-(1-\rho)\mu t} t dt = \frac{1}{\mu(1-\rho)} . \quad (35)$$

2.2 The Case of an Infinite Number of Channels

Although this simple model is useful in bringing to light the types of results which can be obtained it is not very realistic from the point of view of a telephone exchange problem. Let us modify it by assuming that instead of one channel we have infinitely many. In this

case the probability of a call being initiated during a small interval of time Δt will still be $\lambda\Delta t + o(\Delta t)$, completely independent of the number of occupied channels, whereas the probability of a call being terminated during a small interval of time Δt will be $n\mu\Delta t + o(\Delta t)$ where n is the number of occupied channels.

Thus if we let $P_n(t)$ be the probability of n occupied channels at time t we obtain the system of equations

$$\frac{dP_n(t)}{dt} = -(\lambda + n\mu)P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t),$$

$$n \geq 1,$$

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t). \quad (36)$$

Proceeding as before, we again wish to determine the generating function

$$P = P(z, t) = \sum_{n=0}^{\infty} P_n(t) z^n, \quad |z| \leq 1, \quad (37)$$

in order to calculate the coefficients of z^n . However, in this case it is somewhat easier as we do not need the Laplace transform.

On multiplying the first equation in (36) by z^n and summing over the appropriate ranges of n , including the second equation, we obtain

$$\begin{aligned}
\frac{\partial P}{\partial t} &= -\lambda P - z\mu \frac{\partial P}{\partial z} + \lambda zP + \mu \frac{\partial P}{\partial z} \\
&= (1-z) \left(-\lambda P + \mu \frac{\partial P}{\partial z} \right)
\end{aligned} \tag{38}$$

or, in standard form ,

$$\frac{\partial P}{\partial t} - (1-z)\mu \frac{\partial P}{\partial z} = -\lambda(1-z)P . \tag{39}$$

The solution, using ordinary methods, is obtained as follows:
equation (39) has the associated Lagrange equations

$$\frac{dt}{1} = \frac{dz}{-(1-z)\mu} = \frac{dP}{-\lambda(1-z)P} . \tag{40}$$

Now $\frac{dt}{1} = \frac{dz}{-(1-z)\mu}$ yields

$$u(z,t,P) \equiv (1-z)e^{-\mu t} = c_1$$

and $\frac{dz}{-(1-z)\mu} = \frac{dP}{-\lambda(1-z)P}$ yields

$$v(z,t,P) \equiv Pe^{-(\lambda/\mu)z} = c_2 .$$

Hence the general solution is given by

$$P = e^{(\lambda/\mu)z} g[(1-z)e^{-\mu t}] . \tag{41}$$

If, at $t = 0$, i channels are busy we have that $P(z, 0) = z^i$

$$\Rightarrow z^i = e^{(\lambda/\mu)z} g[(1-z)e^0]$$

$$\Rightarrow g[(1-z)e^{-\mu t}] = e^{-(\lambda/\mu)(1-(1-z)e^{-\mu t})} (1-(1-z)e^{-\mu t})^i. \quad (42)$$

When this is substituted in (41) we obtain

$$\begin{aligned} P(z, t) &= e^{(\lambda/\mu)z} e^{-(\lambda/\mu)(1-(1-z)e^{-\mu t})} (1-(1-z)e^{-\mu t})^i \\ &= \{\exp [-(\lambda/\mu)(1-z)(1-e^{-\mu t})]\} [1-(1-z)e^{-\mu t}]^i. \end{aligned} \quad (43)$$

Thus we have obtained an expression for $P(z, t)$ and therefore can compute $P_n(t)$ by calculating the coefficient of z^n .

However, as this coefficient is

$$P_n(t) = \frac{1}{n!} \frac{\partial^n P(z, t)}{\partial z^n} \Big|_{z=0}, \quad (44)$$

we obtain

$$P_n(t) = \frac{e^{-(\lambda/\mu)(1-e^{-\mu t})}}{n!} \sum_{k=0}^n \binom{n}{k} (\lambda/\mu)^{n-k} \frac{i!}{(i-k)!} (1-e^{-\mu t})^{n+i-2k} e^{-k\mu t}$$

for $n \leq i$ (45)

and

$$P_n(t) = \frac{e^{-(\lambda/\mu)(1-e^{-\mu t})}}{n!} \sum_{k=0}^i \binom{n}{k} (\lambda/\mu)^{n-k} \frac{i!}{(i-k)!} (1-e^{-\mu t})^{n+i-2k} e^{-k\mu t}$$

for $n > i$ (46)

by applying Leibnitz's differentiation formula

$$\frac{d^n f(z)g(z)}{dz^n} = \sum_{k=0}^n \binom{n}{k} f^{(n-k)}(z) g^{(k)}(z) \quad (47)$$

to $P(z,t)$ with $f(z) = e^{(\lambda/\mu)(1-e^{-\mu t})z}$ and $g(z) = [1-(1-z)e^{-\mu t}]^i$.

In order to find the average number of channels occupied at time t , multiply the first equation of (36) by n and add over the appropriate ranges of n . This gives

$$\sum_{n=1}^{\infty} nP'_n(t) = M'(t) = \lambda - \mu M(t). \quad (48)$$

If we assume at $t = 0$, i channels are busy, then $M(0) = i$ and

$$M(t) = (\lambda/\mu)(1-e^{-\mu t}) + ie^{-\mu t}. \quad (49)$$

It is interesting to note that in the special case $i = 0$ we have

$$P_n(t) = \frac{e^{-(\lambda/\mu)(1-e^{-\mu t})} (\lambda/\mu)^n (1-e^{-\mu t})^n}{n!}, \quad (50)$$

that is, the $P_n(t)$ are given exactly by the Poisson distributions with mean $M(t)$.

To determine the equations for the probabilities p_n which describe the steady state, as before we set the time derivatives equal to zero and eliminate time from the system of equations (36), obtaining

$$\lambda p_0 = \mu p_1,$$

$$(\lambda + n\mu)p_n = \lambda p_{n-1} + (n+1)\mu p_{n+1}, \quad n \geq 1, \quad (51)$$

which gives by induction

$$\begin{aligned} p_n &= p_0 (\lambda/\mu)^n / n! \\ \Rightarrow p_n &= \frac{e^{-(\lambda/\mu)} (\lambda/\mu)^n}{n!}. \end{aligned} \quad (52)$$

Thus the limiting distribution is a Poisson distribution with parameter λ/μ . It is independent of the initial state.

2.3 The Case of a Finite Number of Channels—Delay System

In the above example we are not concerned with waiting lines as we have an infinite number of channels. However, the situation is still somewhat unrealistic. If we assume instead that we have a finite number of channels, say c , and that if all channels are busy a single waiting line forms and calls enter the channels when they are vacant on a first-come-first-served basis, we will approximate the situation of a long distance switchboard.

Again by letting $P_n(t)$ be the probability of n calls in the system at time t we obtain the system of equations

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t),$$

$$\frac{dP_n(t)}{dt} = -(\lambda + n\mu)P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t), \quad (53)$$

$$1 \leq n < c,$$

$$\frac{dP_n(t)}{dt} = -(\lambda + c\mu)P_n(t) + \lambda P_{n-1}(t) + c\mu P_{n+1}(t),$$

$$c \leq n.$$

We begin the solution of these equations in much the same manner as those in Erlang's model by considering the generating function

$$P = P(z, t) = \sum_{n=0}^{\infty} P_n(t) z^n, \quad |z| \leq 1, \quad (54)$$

and taking the Laplace transform.

Rewriting the system (53) in the form

$$\begin{aligned}
 P'_0(t) &= -(\lambda + c\mu)P_0(t) + c\mu P_0(t) + c\mu P_1(t) - (c-1)\mu P_1(t), \\
 P'_n(t) &= -(\lambda + c\mu)P_n(t) + (c-n)\mu P_n(t) + \lambda P_{n-1}(t) \\
 &\quad + c\mu P_{n+1}(t) - (c-n-1)\mu P_{n+1}(t), \quad 1 \leq n < c, \\
 P'_n(t) &= -(\lambda + c\mu)P_n(t) + \lambda P_{n-1}(t) + c\mu P_{n+1}(t), \quad c \leq n,
 \end{aligned} \tag{55}$$

multiplying each equation of (55) by z^n and summing gives

$$\begin{aligned}
 \frac{\partial P}{\partial t} &= -(\lambda + c\mu)P + \lambda zP + c\mu \frac{P - P_0(t)}{z} + c\mu P_0(t) \\
 &\quad - (c-1)\mu(1-z)P_1(t) - (c-2)\mu z(1-z)P_2(t) - \dots \\
 &\quad - (c-n)\mu z^{n-1}(1-z)P_n(t) - \dots - \mu z^{c-2}(1-z)P_{c-1}(t).
 \end{aligned} \tag{56}$$

Taking the Laplace transform of this expression and solving for $P^*(z, s)$ gives

$$P^*(z, s) = \frac{z^{i+1} - (1-z)\mu \sum_{n=0}^{c-1} (c-n)z^n P_n^*(s)}{sz - (1-z)(c\mu - \lambda z)} \tag{57}$$

if we assume $P_i(0) = 1$.

Since $P^*(z,s)$ converges inside and on the unit circle for $\text{Re}(s) > 0$, the zeros of the numerator must correspond with those of the denominator in this domain.

The zeros of the denominator are

$$\alpha_k(s) = \frac{\lambda + c\mu + s \pm [(\lambda + c\mu + s)^2 - 4\lambda c\mu]^{\frac{1}{2}}}{2\lambda}, \quad k = 1, 2, \quad (58)$$

where $\alpha_1(s)$ has the positive sign before the radical.

Again, applying Rouché's Theorem to the denominator $sz - (1-z)(c\mu - \lambda z)$, we find that $\alpha_2(s)$ is its only zero inside the unit circle and thus the numerator $z^{i+1} - \mu(1-z) \sum_{n=0}^{c-1} (c-n)z^n P_n^*(s)$ must vanish for $z = \alpha_2$, that is

$$\sum_{n=0}^{c-1} (c-n) \alpha_2^n P_n^*(s) = \frac{\alpha_2^{i+1}}{\mu(1-\alpha_2)}. \quad (59)$$

Now in order to obtain $P_n^*(s)$, and thus $P_n(t)$, for $0 \leq n \leq c-1$ we require c equations involving the c unknowns $P_n^*(s)$, $0 \leq n \leq c-1$. These c equations are (59) and the first $c-1$ equations of (53) which are

$$P'_0(t) = -\lambda P_0(t) + \mu P_1(t),$$

$$P'_n(t) = -(\lambda + n\mu)P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t), \quad (60)$$

$$1 \leq n \leq c-2.$$

To solve these equations we introduce the generating function

$$Q(z,t) = \sum_{n=0}^{c-2} P_n(t) z^n . \quad (61)$$

Multiplying the second equation of (60) by z^n and summing gives

$$\frac{\partial Q}{\partial t} = -\lambda Q - \mu z \frac{\partial Q}{\partial z} + \mu \frac{\partial Q}{\partial z} + \lambda z Q - \lambda z^{c-1} P_{c-2} + \mu (c-1) z^{c-2} P_{c-1} \quad (62)$$

$$\Rightarrow -\frac{\partial Q}{\partial t} + \mu (1-z) \frac{\partial Q}{\partial z} = \lambda (1-z) Q + \lambda z^{c-1} P_{c-2} - \mu (c-1) z^{c-2} P_{c-1} \quad (63)$$

$$\Rightarrow \frac{dt}{-1} = \frac{dz}{\mu (1-z)} = \frac{dQ}{\lambda (1-z) Q + \lambda z^{c-1} P_{c-2} - \mu (c-1) z^{c-2} P_{c-1}} . \quad (64)$$

The first equation gives

$$c_1 e^{\mu t} = (1-z) \Rightarrow c_1 = (1-z) e^{-\mu t} \Rightarrow z = 1 - c_1 e^{\mu t} . \quad (65)$$

Using this value of z in the third member and taking the equation in the first and third members gives

$$\frac{\partial Q}{\partial t} + \lambda c_1 e^{\mu t} Q = \mu (c-1) (1-c_1 e^{\mu t})^{c-2} P_{c-1} - \lambda (1-c_1 e^{\mu t})^{c-1} P_{c-2} \quad (66)$$

with the solution

$$Q = e^{-(\lambda/\mu)c_1 e^{\mu t}} \left\{ \int_0^t [\mu(c-1)(1-c_1 e^{\mu x})^{c-2} P_{c-1}(x) - \lambda(1-c_1 e^{\mu x})^{c-1} P_{c-2}(x)] \cdot e^{(\lambda/\mu)c_1 e^{\mu x}} dx + c_2 \right\} . \quad (67)$$

In order to obtain $c_2 = f(c_1)$ we must consider two cases.

First of all if the initial number in the system $i \leq c - 2$,

$$Q(z, 0) = z^i$$

$$\Rightarrow f(1-z) = z^i e^{(\lambda/\mu)(1-z)} \quad (68)$$

$$\text{as } f(c_1) = f[(1-z)e^{-\mu t}]$$

$$\Rightarrow f[(1-z)e^{-\mu t}] = [1-(1-z)e^{-\mu t}]^i \exp [(\lambda/\mu)(1-z)e^{-\mu t}] \quad (69)$$

$$\begin{aligned} \Rightarrow Q(z, t) = e^{-(\lambda/\mu)(1-z)} \int_0^t \{ & \mu(c-1)[1-(1-z)e^{-\mu(t-x)}]^{c-2} P_{c-1}(x) \\ & - \lambda[1-(1-z)e^{-\mu(t-x)}]^{c-1} P_{c-2}(x) \} \\ & \exp [(\lambda/\mu)(1-z)e^{-\mu(t-x)}] dx \\ & + \exp [-(\lambda/\mu)(1-z)(1-e^{-\mu t})][1-(1-z)e^{-\mu t}]^i . \end{aligned} \quad (70)$$

Secondly, if the initial number in the system $i > c - 2$,

$$Q(z, 0) = 0 \quad \text{and hence} \quad c_2 = 0 .$$

Continuing the problem for $i > c - 2$, we use the result³

³Erdelyi, A. (ea.): "Tables of Integral Transforms", vol. I, McGraw Hill Book Co., Inc., New York, 1954, page 147, (40).

$$\begin{aligned}
& \int_0^{\infty} e^{-sw} [1 - (1-z)e^{-\mu w}]^{c-2} e^{(\lambda/\mu)(1-z)e^{-\mu w}} dw \\
&= \int_0^{\infty} e^{-(s/\mu)y} [1 - (1-z)e^{-y}]^{c-2} e^{(\lambda/\mu)(1-z)e^{-y}} \frac{dy}{\mu} \\
&= \frac{1}{\mu} \frac{\Gamma(s/\mu)}{\Gamma((s/\mu)+1)} \phi_1\left[\frac{s}{\mu}, -(c-2), 1, (1-z), (\lambda/\mu)(1-z)\right]
\end{aligned}$$

where

$$\phi_1(\alpha, \beta, \gamma, x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_n}{(\gamma)_{m+nm} n!} x^m y^n,$$

$$(\alpha)_0 = 1 \quad \text{and} \quad (\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1).$$

This gives

$$\begin{aligned}
Q^*(z, s) &= e^{-(\lambda/\mu)(1-z)} \left\{ (c-1) P_{c-1}^*(s) \frac{\Gamma(s/\mu)}{\Gamma((s/\mu)+1)} \right. \\
&\quad \phi_1[s/\mu, -(c-2), 1, (1-z), (\lambda/\mu)(1-z)] - (\lambda/\mu) P_{c-2}^*(s) \frac{\Gamma(s/\mu)}{\Gamma((s/\mu)+1)} \\
&\quad \left. \phi_1[s/\mu, -(c-1), 1, (1-z), (\lambda/\mu)(1-z)] \right\} \tag{71}
\end{aligned}$$

as the first expression of (70) is a convolution.

We have

$$P_{c-2}^*(s) = \frac{1}{(c-2)!} \frac{\partial^{c-2} Q^*}{\partial z^{c-2}} \Big|_{z=0} \tag{72}$$

$$\text{as } Q^*(z, s) = \sum_{n=0}^{c-2} P_n^*(s) z^n .$$

Letting $f(z) = e^{-(\lambda/\mu)(1-z)}$ and $g(z) = \phi_1[s/\mu, -(c-2), 1, (1-z), (\lambda/\mu)(1-z)]$, applying the Leibnitz's differentiation theorem gives

$$\begin{aligned} & \frac{1}{(c-2)!} \frac{\partial^{c-2} f(z) g(z)}{\partial z^{c-2}} \Big|_{z=0} \\ &= \frac{1}{(c-2)!} \sum_{k=0}^{c-2} \binom{c-2}{k} (\lambda/\mu)^{c-k-2} e^{-\lambda/\mu} \frac{d^k}{dz^k} \phi_1\left[\frac{s}{\mu}, -(c-2), 1, (1-z), (\lambda/\mu)(1-z)\right] \Big|_{z=0} . \end{aligned} \quad (73)$$

Thus

$$\begin{aligned} P_{c-2}^*(s) &= \frac{\Gamma(s/\mu)}{(c-2)! \Gamma((s/\mu)+1)} \left\{ (c-1) P_{c-1}^*(s) \sum_{k=0}^{c-2} \binom{c-2}{k} \right. \\ & \quad (\lambda/\mu)^{c-k-2} e^{-\lambda/\mu} \frac{d^k}{dz^k} \phi_1[s/\mu, -(c-2), 1, (1-z), (\lambda/\mu)(1-z)] \Big|_{z=0} \\ & \quad - (\lambda/\mu) P_{c-2}^*(s) \sum_{k=0}^{c-2} \binom{c-2}{k} (\lambda/\mu)^{c-k-2} e^{-\lambda/\mu} \\ & \quad \left. \frac{d^k}{dz^k} \phi_1[s/\mu, -(c-1), 1, (1-z), (\lambda/\mu)(1-z)] \Big|_{z=0} \right\} . \end{aligned} \quad (74)$$

Solving for $P_{c-2}^*(s)$ yields

$$\begin{aligned}
P_{c-2}^*(s) &= \left\{ \frac{\Gamma(s/\mu)(c-1)}{(c-2)! \Gamma((s/\mu)+1)} P_{c-1}^*(s) \sum_{k=0}^{c-2} \binom{c-2}{k} (\lambda/\mu)^{c-k-2} e^{-\lambda/\mu} \right. \\
&\quad \left. \frac{d^k}{dz^k} \phi_1[s/\mu, -(c-2), 1, (1-z), (\lambda/\mu)(1-z)] \Big|_{z=0} \right\} \div \\
&\quad \left\{ 1 + (\lambda/\mu) \frac{\Gamma(s/\mu)}{(c-2)! \Gamma((s/\mu)+1)} \sum_{k=0}^{c-2} \binom{c-2}{k} (\lambda/\mu)^{c-k-2} e^{-\lambda/\mu} \right. \\
&\quad \left. \frac{d^k}{dz^k} \phi_1[s/\mu, -(c-1), 1, (1-z), (\lambda/\mu)(1-z)] \Big|_{z=0} \right\}. \quad (75)
\end{aligned}$$

We also have in exactly the same manner as above for $n < c - 2$

$$\begin{aligned}
P_n^*(s) &= \frac{\Gamma(s/\mu)}{n! \Gamma((s/\mu)+1)} \left\{ (c-1) P_{c-1}^*(s) \sum_{k=0}^n \binom{n}{k} (\lambda/\mu)^{n-k} e^{-\lambda/\mu} \right. \\
&\quad \left. \frac{d^k}{dz^k} \phi_1[s/\mu, -(c-2), 1, (1-z), (\lambda/\mu)(1-z)] \Big|_{z=0} \right. \\
&\quad \left. - (\lambda/\mu) P_{c-2}^*(s) \sum_{k=0}^n \binom{n}{k} (\lambda/\mu)^{n-k} e^{-\lambda/\mu} \right. \\
&\quad \left. \frac{d^k}{dz^k} \phi_1[s/\mu, -(c-1), 1, (1-z), (\lambda/\mu)(1-z)] \Big|_{z=0} \right\}. \quad (76)
\end{aligned}$$

Thus as

$$\sum_{n=0}^{c-1} (c-n) \alpha_2^n P_n^*(s) = \frac{\alpha_2^{i+1}}{\mu(1-\alpha_2)} \quad (77)$$

we have

$$P_{c-1}^*(s) = \frac{\alpha_2^{i+1}}{\mu(1-\alpha_2)} \div \left\{ \alpha_2^{c-1} + \sum_{n=0}^{c-2} (c-n) \alpha_2^n \frac{P_n^*(s)}{P_{c-1}^*(s)} \right\} \quad (78)$$

and thus we have our c equations involving the c unknowns P_n^* ,
 $0 \leq n \leq c-1$.

In order to determine $P_n^*(s)$, $n \geq c$, we will again consider
the generating function

$$P^*(z,s) = \frac{z^{i+1} - \mu(1-z) \sum_{n=0}^{c-1} (c-n) z^n P_n^*(s)}{sz - (1-z)(c\mu - \lambda z)} \quad (79)$$

and determine the coefficients of z^n . To do this write the denominator
in the form $-\lambda(z - \alpha_1)(z - \alpha_2)$ and apply Leibnitz's theorem to the
product $(z - \alpha_1)^{-1}(z - \alpha_2)^{-1}$. We have

$$\begin{aligned} & \frac{1}{m!} \frac{d^m}{dz^m} (z - \alpha_1)^{-1} (z - \alpha_2)^{-1} \Big|_{z=0} \\ &= \frac{1}{m!} \sum_{k=0}^m \binom{m}{k} (-1)^k k! (-\alpha_1)^{-(k+1)} (-1)^{m-k} (m-k)! (-\alpha_2)^{-(m-k+1)} \end{aligned} \quad (80)$$

$$\begin{aligned} &= \sum_{k=0}^m (\alpha_1)^{-(k+1)} (\alpha_2)^{-(m-k+1)} \\ &= \sum_{k=0}^m \alpha_1^{-1} \alpha_2^{-m-1} (\alpha_2/\alpha_1)^k \\ &= \frac{\lambda}{c\mu\alpha_2^m} \cdot \frac{1 - (\alpha_2/\alpha_1)^{m+1}}{1 - (\alpha_2/\alpha_1)} . \end{aligned} \quad (81)$$

Now

$$P_n^*(s) = \frac{1}{n!} \frac{\partial^n}{\partial z^n} P^*(z, s) \Big|_{z=0} \quad (82)$$

and thus

$$P_n^*(s) = \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \frac{d^k A(z)}{dz^k} \frac{d^{n-k} B(z)}{dz^{n-k}} \Big|_{z=0} \quad (83)$$

where $A(z) = z^{i+1} - \mu \sum_{j=0}^{c-1} (c-j) z^j P_j^*(s) + \mu \sum_{j=0}^{c-1} (c-j) z^{j+1} P_j^*(s)$ and

$B(z) = (-1/\lambda)(z-\alpha_1)^{-1}(z-\alpha_2)^{-1}$ as $\alpha_1 + \alpha_2 = \frac{\lambda+c\mu+s}{\lambda}$, $\alpha_1\alpha_2 = \frac{c\mu}{\lambda}$ and $s = -\lambda(1-\alpha_1)(1-\alpha_2)$,

$$\begin{aligned} \Rightarrow P_n^*(s) &= \frac{-\mu}{n!} \sum_{j=0}^{c-1} \binom{n}{j} (c-j) j! P_j^*(s) \frac{d^{n-j} B(z)}{dz^{n-j}} \Big|_{z=0} \\ &+ \frac{\mu}{n!} \sum_{j=0}^{c-1} \binom{n}{j+1} (c-j) (j+1)! P_j^*(s) \frac{d^{n-(j+1)} B(z)}{dz^{n-(j+1)}} \Big|_{z=0} \\ &+ \frac{1}{(n-i-1)!} \cdot \frac{d^{n-(i+1)} B(z)}{dz^{n-(i+1)}} \Big|_{z=0} \end{aligned} \quad (84)$$

$$\begin{aligned} &= (\mu/\lambda) \sum_{j=0}^{c-1} [(c-j) P_j^*(s) \frac{\lambda}{c\mu\alpha_2^{n-j}} \frac{1-(\alpha_2/\alpha_1)^{n-j+1}}{1-\alpha_2/\alpha_1} \\ &- (c-j) P_j^*(s) \frac{\lambda}{c\mu\alpha_2^{n-j-1}} \frac{1-(\alpha_2/\alpha_1)^{n-j}}{1-\alpha_2/\alpha_1}] \\ &- \frac{1}{c\mu\alpha_2^{n-i-1}} \frac{1-(\alpha_2/\alpha_1)^{n-i}}{1-\alpha_2/\alpha_1} \quad \text{for } n \geq c. \end{aligned} \quad (85)$$

Thus we have obtained $P_n^*(s)$ for $n = 0, 1, 2, \dots$ and now may obtain $P_n(t)$ for $n = 0, 1, 2, \dots$ by considering the inverse transform.

In the same way as before we obtain the equations describing the steady state as

$$\begin{aligned}\lambda p_0 &= \mu p_1, \\ (\lambda + n\mu)p_n &= \lambda p_{n-1} + (n+1)\mu p_{n+1}, \quad 1 \leq n < c, \\ (\lambda + c\mu)p_n &= \lambda p_{n-1} + c\mu p_{n+1}, \quad n \geq c.\end{aligned}\tag{86}$$

By induction we find that

$$\begin{aligned}p_n &= p_0 \frac{(\lambda/\mu)^n}{n!}, \quad n \leq c, \\ p_n &= p_0 \frac{(\lambda/\mu)^n}{c! c^{n-c}} = \frac{p_0 (\lambda/c\mu)^n c^c}{c!}, \quad n \geq c.\end{aligned}\tag{87}$$

However, the series $\sum \frac{p_n}{p_0}$ converges only if $\lambda/\mu < c$.

Hence a limiting distribution cannot exist when $\lambda \geq c\mu$. In this case $p_n = 0$ for all n , which means that gradually the waiting line grows over all bounds.

On the other hand, if $\lambda/\mu < c$, using $\sum_{n=0}^{\infty} p_n = 1$, we obtain

$$1 = p_0 \left[\sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} + \sum_{n=c}^{\infty} \frac{\rho^n c^c}{c!} \right]\tag{88}$$

$$\text{where } \rho = \frac{\lambda}{c\mu} ,$$

$$\Rightarrow p_0 = \frac{1}{\sum_{n=0}^{c-1} \frac{(c\rho)^n}{n!} + \frac{(c\rho)^c}{c!(1-\rho)}} . \quad (89)$$

The expected number in the system is given by

$$L = \sum_{n=0}^{\infty} n p_n = p_0 \sum_{n=0}^c \frac{n(\lambda/\mu)^n}{n!} + p_0 \sum_{n=c+1}^{\infty} n \frac{(\lambda/c\mu)^n c^n}{c!} \quad (90)$$

$$= c\rho + \frac{\rho (c\rho)^c}{c!(1-\rho)^2} p_0 \quad (91)$$

whereas the expected number in the queue is

$$L_q = \sum_{n=c+1}^{\infty} (n-c) p_n = \frac{\rho (c\rho)^c}{c!(1-\rho)^2} p_0 . \quad (92)$$

Again in this example the situation may occur where an arrival will have to wait. The probability of this occurring is clearly the probability that c or more channels are occupied which is

$$\sum_{n=c}^{\infty} \frac{p_0}{c!} (\rho)^n c^c = \frac{p_0 c^c}{c!} \cdot \frac{\rho^c}{1-\rho} = \frac{p_0 (c\rho)^c}{c!(1-\rho)} . \quad (93)$$

To determine the probability $P(T>t)$ of waiting longer than time t consider the probability of the n th customer in the line

waiting longer than time t . This is

$$P\{T_1 + T_2 + \dots + T_n > t\} \quad (94)$$

where T_i is a random variable giving the amount of time all c channels are occupied before any of them becomes free.

Thus $P\{T_i > t\}$ = the probability that none of the c channels become free in time t which is $e^{-c\mu t}$ and therefore T_i has an exponential distribution with mean $c\mu$.

The characteristic function of this exponential distribution is $\frac{c\mu}{c\mu - i\phi}$ and thus the characteristic function of the distribution of the waiting time of the n th customer in the line is $(\frac{c\mu}{c\mu - i\phi})^n$, thus giving a density of

$$\frac{c\mu (c\mu t)^{n-1} e^{-c\mu t}}{(n-1)!} \quad (95)$$

which implies

$W_n(t)$ = the probability that the n th customer in the line is served by time t

$$= \int_0^t \frac{(c\mu) (c\mu x)^{n-1} e^{-c\mu x}}{(n-1)!} dx. \quad (96)$$

The density function of the distribution of the waiting time of any customer in the line is given by

$$\begin{aligned}
& \sum_{n=1}^{\infty} p_{c+(n-1)} \frac{(c\mu) (c\mu t)^{n-1} e^{-c\mu t}}{(n-1)!} \\
&= \frac{p_o c^c}{c!} \sum_{n=1}^{\infty} (\rho)^{c+(n-1)} \frac{(c\mu) (c\mu t)^{n-1} e^{-c\mu t}}{(n-1)!} \\
&= \frac{p_o c^c (c\mu) e^{-c\mu t} \rho^c}{c!} \sum_{n=1}^{\infty} \frac{(\rho c\mu t)^{n-1}}{(n-1)!} \\
&= \frac{p_o c^c (c\mu) \rho^c}{c!} e^{-c\mu t (1-\rho)} \quad (97)
\end{aligned}$$

and thus we can calculate the probability that an arriving customer will wait longer than time t as

$$\begin{aligned}
P(T > t) &= \frac{p_o c^c (c\mu) \rho^c}{c!} \int_t^{\infty} e^{-c\mu x (1-\rho)} dx \\
&= \frac{p_o c^c \rho^c}{c! (1-\rho)} e^{-c\mu t (1-\rho)} \\
&= e^{-c\mu t (1-\rho)} P(T > 0) \quad (98)
\end{aligned}$$

Approaching this problem from a different point of view, we clearly have $W_1(t)$ equal to the probability that any of the c channels completes service in time t and thus

$$\begin{aligned}
W_n(t) &= \int_0^t W_{n-1}(t-x) d_x W_1(x) \\
&= 1 - e^{-c\mu t} \sum_{i=0}^{n-1} \frac{(c\mu t)^i}{i!} \quad (99)
\end{aligned}$$

by induction.

The probability of waiting less than time t is thus

$$P(< t) = \sum_{n=0}^{c-1} p_n + \sum_{n=1}^{\infty} p_{c+(n-1)} W_n(t) \quad (100)$$

where $W_n(t)$ is given above.

2.4 The Case of a Finite Number of Channels—Loss System

In concluding this first part we alter the delay system just discussed by assuming that a call is completely lost to the system if all c channels are busy (that is, no waiting line is formed). This is the situation of the trunks that are available to a long-distance operator when she places a call for a customer or those available to a certain group of subscribers for the use of their telephones.

Again by letting $P_n(t)$ be the probability of n calls in the system at time t , we obtain the system of equations

$$P'_0(t) = -\lambda P_0(t) + \mu P_1(t) ,$$

$$P'_n(t) = -(\lambda + n\mu)P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t) ,$$

$$1 \leq n \leq c-1 ,$$

$$P'_c(t) = -c\mu P_c(t) + \lambda P_{c-1}(t) . \quad (101)$$

These differ from the previous situation only in the last equation.

The equations describing the steady state are

$$\begin{aligned}\lambda p_0 &= \mu p_1, \\ (\lambda + n\mu)p_n &= \lambda p_{n-1} + (n+1)\mu p_{n+1}, \quad 1 \leq n \leq c-1, \\ c\mu p_c &= \lambda p_{c-1},\end{aligned}\tag{102}$$

and as before we have

$$p_n = \frac{p_0 (\lambda/\mu)^n}{n!}.\tag{103}$$

Since $\sum_{n=0}^c p_n = 1$,

$$p_0 \sum_{n=0}^c \frac{(\lambda/\mu)^n}{n!} = 1,\tag{104}$$

which implies

$$p_0 = \frac{1}{1 + \lambda/\mu + (\lambda/\mu)^2/2! + \dots + (\lambda/\mu)^c/c!}\tag{105}$$

and thus

$$p_n = \frac{\rho^n/n!}{1 + \rho + \rho^2/2! + \dots + \rho^c/c!}\tag{106}$$

where $\rho = \lambda/\mu$.

Since the probability that a call is lost equals the probability that all c channels are busy we have the probability of a lost call as

$$p_c = \frac{\rho^c / c!}{1 + \rho + \rho^2 / 2! + \dots + \rho^c / c!} \quad (107)$$

which is called the Erlang loss formula.

As $c \rightarrow \infty$ we have the case of an infinite number of channels. Letting $c \rightarrow \infty$ in (106) we obtain the Poisson distribution

$$\begin{aligned} p_n &= \frac{\rho^n / n!}{1 + \rho + \rho^2 / 2! + \dots + \rho^n / n! + \dots} \\ &= \frac{\rho^n}{n!} e^{-\rho}, \quad \rho = \lambda / \mu, \end{aligned} \quad (108)$$

which agrees with our previous result.

As clearly there are no waiting times in this situation, we have completed the first part of the thesis and are ready to begin the analysis of the telephone usage of a group of local subscribers.

CHAPTER III

AN ACTUAL TELEPHONE EXCHANGE SITUATION

3.1 A Description of the Data

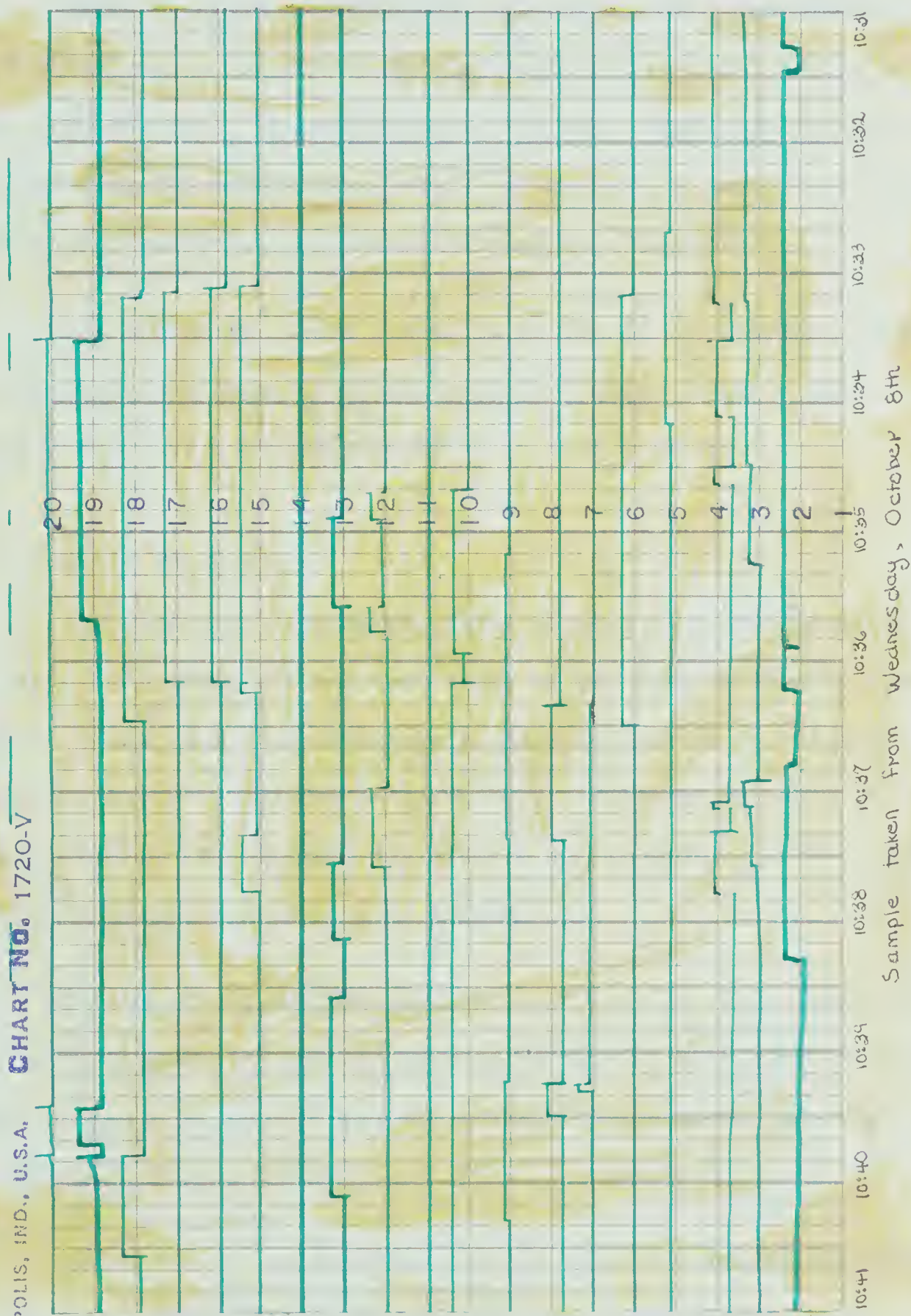
The data that have been used in this analysis were recorded by a pen recorder chart on the calls terminating at the 699 77— Sherwood Park telephones from 9 A.M. to 5 P.M. over a period of about eight days. The actual days and times considered are as follows:

Thursday, October 2nd, 1969	9:00 A.M. to 5:00 P.M.
Friday, October 3rd, 1969	9:00 A.M. to 5:00 P.M.
Monday, October 6th, 1969	9:00 A.M. to 5:00 P.M.
Tuesday, October 7th, 1969	9:00 A.M. to 10:39 A.M. and 11:00 A.M. to 5:00 P.M.
Wednesday, October 8th, 1969	9:00 A.M. to 5:00 P.M.
Thursday, October 9th, 1969	9:00 A.M. to 5:00 P.M.
Friday, October 10th, 1969	9:00 A.M. to 12:00 P.M. and 2:30 P.M. to 5:00 P.M.
Tuesday, October 14th, 1969	9:00 A.M. to 1:00 P.M.

Since the 699 77— telephones are all business phones which operate on a rotary hunting principle, the times and days considered give a reasonably accurate description of their use.

When one dials a number beginning with 699 77— it is necessary to make a connection through one of the ten trunks represented by the first ten lines (lines 1 to 10) of the pen recorded chart. (See the example on page 44). At the instant when the fifth digit in the number has been dialed, a connection is made with one of the ten trunks and the pen corresponding to that trunk will jump upward, remaining in its new position until the completion of the call. A call that is attempted when all channels are busy is lost. Clearly, we have the fourth situation discussed in Chapter II - that of a finite number of

Sample of the Pen Recorder Chart



channels available to a group of subscribers where a call is lost if all channels are occupied.

Thus, by again assuming that all calls arrive independently of each other and of the number in the system and that the length of each call is independent of every other call, we can compare the distributions calculated from the interarrival times and the busy times of the given data with the exponential situation discussed in Chapter II.

The second ten lines (lines 11 to 20) on the chart show both the originating and terminating calls on particular Sherwood Park telephones and have not been considered in this analysis.

3.2 The Method of Analysis

The data were considered, first of all, on an hourly basis and the number of interarrival and busy times of the various lengths were recorded. Since it was not feasible to measure lengths on the chart more accurately, they were grouped into intervals of length ten seconds, which is represented by the smallest division on the recording. These results are given in Tables 1.1 to 1.16 of Appendix I.

As the object of this analysis is to see if these distributions are actually exponential, it was first necessary to estimate the parameters λ and μ where $\frac{1}{\lambda}$ and $\frac{1}{\mu}$ are the average interarrival and busy times respectively.

Thus, if T_1 is the interarrival time in seconds, we assumed that T_1 has an exponential distribution with parameter λ where λ is unknown. Letting $x_1, x_2, \dots, x_N$ be N independent observations of

T_1 with joint density function

$$\frac{1}{\lambda^N} e^{-\lambda \sum_{i=1}^N x_i} = \frac{1}{\lambda^N} e^{-\lambda N \bar{x}}$$

we have the maximum likelihood equation

$$\frac{d}{d\hat{\lambda}} \{N \ln \hat{\lambda} - \hat{\lambda} N \bar{x}\} = 0 ,$$

which has the unique solution

$$\hat{\lambda} = \frac{1}{\bar{x}} .$$

From the fact that the second derivative of the log-likelihood function

$$\frac{d^2}{d\hat{\lambda}^2} \{N \ln \hat{\lambda} - \hat{\lambda} N \bar{x}\} = \frac{-N}{\hat{\lambda}^2}$$

is everywhere negative, we know that the above solution is the maximum likelihood estimate.

Now, if n_r is the number of observations in $[r, r+1)h$, $r = 0, 1, 2, \dots$, $N_o = \sum_{r \geq 0} n_r$, and $S_o = \sum_{r \geq 0} r n_r$, we have $\bar{X}_1 = (\frac{S_o}{N_o} + \frac{1}{2})$

where $\bar{X}_1 = \frac{T_1}{h}$ and $h =$ the unit of grouping = ten seconds. Thus, for this grouped data,

$$\bar{T}_1 = (\frac{S_o}{N_o} + \frac{1}{2}) \cdot h$$

giving

$$\hat{\lambda} = \left\{ \left(\frac{S_o}{N_o} + \frac{1}{2} \right) \cdot h \right\}^{-1} .$$

However,

$$P\{rh \leq T_1 < (r+1)h\} = \theta_1^r (1 - \theta_1) , \quad r = 0, 1, 2, \dots ,$$

where $\theta_1 = e^{-\lambda h}$, and on substituting the maximum likelihood value of λ we have

$$\hat{\theta}_1 = e^{-\hat{\lambda}h} = \exp - \left(\frac{S_o}{N_o} + \frac{1}{2} \right)^{-1} .$$

The situation was not quite so straightforward when dealing with the busy times T_2 . The problem arises as the busy time shown by the pen recorder is actually the sum of two random variables, X and Y . X gives the time required to dial two digits (approximately five seconds), the ringing time (approximately nine seconds, if we assume two rings) and time necessary to identify oneself and call the desired party to the telephone and Y gives the actual conversation time. We have assumed that Y is exponential but we have no information concerning the distribution of X and thus have assumed it constant and equal to thirty seconds. This choice is completely arbitrary. The actual value may vary greatly from call to call and cause considerable error in the calculations.

Letting

$$Y_a = \frac{T_2^{-ah}}{h} = \frac{Y}{h} , \quad a = 3, \quad h = 10,$$

where the conditional probability density function of Y given $Y \geq 0$ is

$$\begin{aligned} \mu e^{-\mu y}, & \quad y \geq 0, \\ 0 & \quad y < 0, \end{aligned}$$

we have, by the same type of analysis used for the interarrival times, that

$$\bar{Y}_a = \frac{S_a}{N_a} + \frac{1}{2} - a$$

giving

$$\hat{\mu} = \left\{ \left(\frac{S_a}{N_a} + \frac{1}{2} - a \right) \cdot h \right\}^{-1}$$

where $N_a = \sum_{r \geq a} n_r$ and $S_a = \sum_{r \geq a} r n_r$.

Also,

$$P\{r h \leq Y < (r+1)h\} = \theta_2^r (1 - \theta_2), \quad r = 0, 1, 2, \dots,$$

where $\theta_2 = e^{-\mu h}$, giving

$$\hat{\theta}_2 = e^{-\hat{\mu} h} = \exp - \left(\frac{S_a}{N_a} + \frac{1}{2} - a \right)^{-1}.$$

The maximum likelihood estimates of λ and μ for each hour of each day are given in Tables 1.17 and 1.18 of Appendix I. The corresponding estimates of θ_1 and θ_2 are given in Table 1.19 and

1.20 of Appendix I.

After this, the total observed frequencies for each one hour period were calculated and the results tabulated in Tables 2.1 and 2.2 of Appendix II. Again the maximum likelihood estimates of λ and μ and the corresponding θ_1 and θ_2 were obtained. These results are given in Tables 2.3 and 2.4 of Appendix II.

Finally, using these estimates for θ_1 and θ_2 , χ^2 tests were done on the hourly results, facilitated by the use of the computer. These results are given in Table 2.5 of Appendix II.

Here the χ^2 values were obtained for the interarrival times by comparing the observed frequencies n_r with the expected frequencies $\hat{\theta}_1^r (1 - \hat{\theta}_1) N_o$, $r = 0, 1, 2, \dots$, where $N_o = \sum_{r \geq 0} n_r$. The calculations were pooled for

$$r = 9 \text{ and } 10,$$

$$r = 11 \text{ and } 12,$$

$$\text{and } r \geq 13,$$

giving 10 degrees of freedom.

The computing of the χ^2 values for the Y distribution was again more difficult. Here the observed frequencies n_r , $r = 3, 4, 5, \dots$, were compared with the expected frequencies $\hat{\theta}_2^{r'} (1 - \hat{\theta}_2) N_3$ where $r' = r - 3$, $r' = 0, 1, 2, \dots$, and $N_3 = \sum_{r \geq 3} n_r$. The calculations were pooled for

$$r = 18 \text{ and } 19,$$

$$r = 20 \text{ and } 21,$$

$r = 22$ and $23,$
 $r = 24$ and $25,$
 $r = 26$ and $27,$
 $r = 28, 29$ and $30,$
 $r = 31, 32$ and $33,$
 $r = 34, 35, 36$ and $37,$
 and $r \geq 38,$

giving 22 degrees of freedom.

The graphical representations of Appendix III give a further comparison of the observed and expected frequencies.

3.3 Results and Conclusions

a) Throughout this analysis it has been obvious that the results obtained for the interarrival times during the lunch hour, from 12 noon to 1 P.M., differ substantially from the others.

Considering the maximum likelihood estimates given in Table 2.3 of Appendix II, the value of $\hat{\lambda}$ for this time is certainly smaller than the others. This, of course, should be the case as the number of arrivals during the lunch hour is lower and thus the expected interarrival time is greater.

b) The probability of a call being lost is equal to the probability that all of the ten channels are occupied.

Altogether, 57 hours 9 minutes = 3,429 minutes = 20,574 "ten second intervals" were analysed. Of this time, all channels were occupied simultaneously for 80 "ten second intervals".

Thus, on the basis of this data, the probability of a call being lost is

$$\frac{80}{20,574} = .0039 .$$

c) In the second chapter we discussed the quantity $\frac{\lambda}{\mu}$, the traffic intensity. In order to obtain an approximate value of $\frac{\lambda}{\mu}$ for our data, average values of $\hat{\lambda}$ and $\hat{\mu}$ have been calculated from Table 2.3 of Appendix II, giving

$$\begin{aligned} \hat{\lambda} &= .026, & \hat{\mu} &= .0082 \\ \text{and } \frac{\hat{\lambda}}{\hat{\mu}} &= 3.15 . \end{aligned}$$

Since we are dealing with ten channels, the expected number of arrivals at each channel will be $\frac{1}{10}$ of the total expected number, implying that the expected interarrival time at each channel will be ten times greater, that is, $10 \times \frac{1}{\hat{\lambda}}$. Thus the traffic intensity, or the ratio of expected busy time to expected interarrival time, at each channel is .32. This clearly indicates that the probability of any channel being almost constantly busy is small and hence the probability of a call being lost is very small.

d) Finally, we shall consider the distributions obtained for the interarrival times and the busy times.

With ten degrees of freedom, the probability of obtaining a χ^2 value greater than 18 is .05 and a χ^2 value greater than 16 is .10. Thus the χ^2 value obtained for the interarrival time

distributions, which are given in Table 2.5 of Appendix II, clearly indicate the fit of these distributions to the exponential is excellent. Thus, in this case, we can certainly conclude that the assumption of calls arriving according to a Poisson distribution is reasonable.

However, this is not the case when dealing with the busy time distributions. With 22 degrees of freedom, the probability of a χ^2 value greater than 40 is .01 and a χ^2 value greater than 34 is .05. Thus, from the χ^2 values obtained for the Y distributions, which are given in Table 2.5 of Appendix II, we can only accept the 9 to 10 A.M. and the 4 to 5 P.M. distributions at the 1% level of significance. Thus, although Figures 3.9 to 3.16 of Appendix III seem to indicate that these distributions are tending to be exponential, we must conclude that the exponential assumptions of the first two chapters do not hold for our data.

There are two main reasons for this result. First of all, we are attempting to show that the distribution of $Y = T_2 - X$ is exponential when we have no information concerning the distribution of X . We have been forced to assume X is constant, which is certainly not the case, as the value of X may vary considerably from call to call. However, thirty seconds is a large estimate of the value of X and even in this case the fit is not good.

Secondly, the fact that these particular telephones operate on a rotary hunting system can seriously effect the assumptions of independence. With this system, an individual is aware if a call is waiting for him on another line and thus the incoming calls can affect

the conversation length.

In concluding, we should keep in mind that the behavior and habits of the people who are using these telephones cannot always be made to fit the strict set of assumptions necessary to carry out such an investigation.

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APPENDICES

APPENDIX I: Tables of results for each
hour of each day

TABLE 1.1

Observed Frequencies of the Interarrival Times
Between 9 A.M. and 10 A.M. of Each Day Considered

Length in 10ths of a Second = r	Number = n_r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
0-	14	17	27	21	23	16	28	22
1-	14	19	19	20	12	10	12	15
2-	13	12	13	9	10	12	9	16
3-	13	10	7	12	11	9	12	11
4-	5	7	6	6	11	12	9	7
5-	3	7	3	7	8	7	5	4
6-	5	2	4	6	2	5	3	6
7-	3	3	8	2	4	5	4	6
8-	1	0	4	4	4	5	2	2
9-	0	1	0	0	2	2	4	3
10-	2	0	0	0	0	1	2	2
11-	0	2	1	1	0	1	0	1
12-	2	2	1	2	2	1	2	0
13-	2	1	0	0	1	1	0	1
14-	1	1	1	1	2	1	0	0
15-	1	1	0	1	1	0	0	0
16-	2	1	1	1	0	0	1	0
17-	0	0	1	1	0	0	0	0
18-	1	0	0	0	0	0	1	0
19-	0	2	0	0	0	0	0	0
20-	0	0	2	0	0	0	0	0
21-	0	0	0	0	0	0	1	0
22-	0	0	0	0	0	0	0	1
23-	0	0	0	0	0	0	0	0
24-	0	0	0	0	0	0	0	1
25-	0	0	0	0	0	0	0	0
26-	0	0	0	0	0	0	0	0
27-	0	0	0	0	0	0	0	0
28-	0	0	0	0	0	0	0	0
$\geq$ 29-	0	0	0	0	0	0	0	0
	82	88	98	94	93	88	95	98

TABLE 1.2

Observed Frequencies of the Interarrival Times
Between 10 A.M. and 11 A.M. of Each Day Considered

Length in 10ths of a Second = r	Number = n _r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
0-	14	29	19	14	17	8	22	21
1-	19	13	19	13	12	11	18	18
2-	10	14	9	8	14	11	13	18
3-	10	12	15	5	3	5	11	12
4-	9	6	9	4	8	4	10	10
5-	4	4	8	7	6	1	3	6
6-	4	4	8	3	4	4	3	3
7-	3	4	1	5	6	3	3	2
8-	3	3	2	1	2	5	1	4
9-	1	2	1	1	2	4	1	3
10-	0	1	4	1	4	0	3	1
11-	2	1	2	1	2	1	1	0
12-	1	1	0	0	0	2	1	0
13-	1	0	0	0	2	0	0	0
14-	1	1	0	0	1	1	1	1
15-	0	0	0	0	0	2	1	0
16-	1	2	1	0	0	0	0	0
17-	1	1	0	0	0	0	0	0
18-	0	0	0	1	0	1	0	0
19-	0	0	0	0	0	1	1	0
20-	0	0	0	0	0	0	1	0
21-	0	0	0	0	0	0	0	0
22-	0	0	0	0	0	0	0	0
23-	0	0	0	0	0	0	0	0
24-	0	0	0	0	0	0	0	0
25-	0	0	0	0	0	0	0	0
26-	0	0	0	0	0	0	0	0
27-	1	0	0	0	0	1	0	0
28-	0	0	0	0	0	0	0	0
≥ 29-	0	0	0	0	0	0	0	1
	85	98	98	64	83	65	94	100

Preserved Proportions of the International System

Proportion in							
10ths of a							
Number = n							
Oct. 14th	Oct. 10th	Oct. 6th	Oct. 2nd	Oct. 1st	Oct. 1st	Oct. 1st	Oct. 1st
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9
10	10	10	10	10	10	10	10
11	11	11	11	11	11	11	11
12	12	12	12	12	12	12	12
13	13	13	13	13	13	13	13
14	14	14	14	14	14	14	14
15	15	15	15	15	15	15	15
16	16	16	16	16	16	16	16
17	17	17	17	17	17	17	17
18	18	18	18	18	18	18	18
19	19	19	19	19	19	19	19
20	20	20	20	20	20	20	20
21	21	21	21	21	21	21	21
22	22	22	22	22	22	22	22
23	23	23	23	23	23	23	23
24	24	24	24	24	24	24	24
25	25	25	25	25	25	25	25
26	26	26	26	26	26	26	26
27	27	27	27	27	27	27	27
28	28	28	28	28	28	28	28
29	29	29	29	29	29	29	29
30	30	30	30	30	30	30	30
31	31	31	31	31	31	31	31
32	32	32	32	32	32	32	32
33	33	33	33	33	33	33	33
34	34	34	34	34	34	34	34
35	35	35	35	35	35	35	35
36	36	36	36	36	36	36	36
37	37	37	37	37	37	37	37
38	38	38	38	38	38	38	38
39	39	39	39	39	39	39	39
40	40	40	40	40	40	40	40
41	41	41	41	41	41	41	41
42	42	42	42	42	42	42	42
43	43	43	43	43	43	43	43
44	44	44	44	44	44	44	44
45	45	45	45	45	45	45	45
46	46	46	46	46	46	46	46
47	47	47	47	47	47	47	47
48	48	48	48	48	48	48	48
49	49	49	49	49	49	49	49
50	50	50	50	50	50	50	50
51	51	51	51	51	51	51	51
52	52	52	52	52	52	52	52
53	53	53	53	53	53	53	53
54	54	54	54	54	54	54	54
55	55	55	55	55	55	55	55
56	56	56	56	56	56	56	56
57	57	57	57	57	57	57	57
58	58	58	58	58	58	58	58
59	59	59	59	59	59	59	59
60	60	60	60	60	60	60	60
61	61	61	61	61	61	61	61
62	62	62	62	62	62	62	62
63	63	63	63	63	63	63	63
64	64	64	64	64	64	64	64
65	65	65	65	65	65	65	65
66	66	66	66	66	66	66	66
67	67	67	67	67	67	67	67
68	68	68	68	68	68	68	68
69	69	69	69	69	69	69	69
70	70	70	70	70	70	70	70
71	71	71	71	71	71	71	71
72	72	72	72	72	72	72	72
73	73	73	73	73	73	73	73
74	74	74	74	74	74	74	74
75	75	75	75	75	75	75	75
76	76	76	76	76	76	76	76
77	77	77	77	77	77	77	77
78	78	78	78	78	78	78	78
79	79	79	79	79	79	79	79
80	80	80	80	80	80	80	80
81	81	81	81	81	81	81	81
82	82	82	82	82	82	82	82
83	83	83	83	83	83	83	83
84	84	84	84	84	84	84	84
85	85	85	85	85	85	85	85
86	86	86	86	86	86	86	86
87	87	87	87	87	87	87	87
88	88	88	88	88	88	88	88
89	89	89	89	89	89	89	89
90	90	90	90	90	90	90	90
91	91	91	91	91	91	91	91
92	92	92	92	92	92	92	92
93	93	93	93	93	93	93	93
94	94	94	94	94	94	94	94
95	95	95	95	95	95	95	95
96	96	96	96	96	96	96	96
97	97	97	97	97	97	97	97
98	98	98	98	98	98	98	98
99	99	99	99	99	99	99	99
100	100	100	100	100	100	100	100

TABLE 1.3

Observed Frequencies of the Interarrival Times
Between 11 A.M. and 12 A.M. of Each Day Considered

Length in 10ths of a Second = r	Number = n_r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
0-	17	12	15	25	19	19	19	18
1-	12	11	13	19	19	19	15	16
2-	10	14	12	19	14	9	11	15
3-	11	10	11	8	7	8	8	7
4-	9	2	3	10	7	6	8	3
5-	7	4	2	5	5	5	4	2
6-	6	4	5	6	4	5	7	2
7-	6	1	8	2	1	4	3	3
8-	4	5	4	0	6	3	2	1
9-	3	2	0	3	1	2	3	0
10-	0	1	2	2	4	2	1	3
11-	1	3	0	1	1	2	0	1
12-	0	1	0	1	1	1	1	2
13-	0	0	0	1	1	0	0	2
14-	1	1	2	0	0	1	1	0
15-	0	0	2	1	1	0	0	1
16-	0	0	1	0	0	1	0	0
17-	0	0	0	0	0	0	0	0
18-	1	0	0	1	0	0	0	0
19-	0	1	0	0	1	0	0	0
20-	0	0	0	0	0	1	0	0
21-	0	0	0	0	0	0	0	1
22-	0	0	0	0	0	0	0	1
23-	0	0	1	0	0	0	0	0
24-	0	0	0	0	0	0	0	0
25-	0	0	0	0	0	0	1	0
26-	0	0	0	0	0	0	0	0
27-	0	0	0	0	0	0	0	0
28-	0	0	0	0	0	0	0	0
> 29-	1	2	0	0	0	0	1	1
	89	74	81	104	92	88	85	79

TABLE 1
 Summary of the results of the investigation of the
 factors influencing the rate of the reaction

Length in Tons of a							
Number = 1							
1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9
10	10	10	10	10	10	10	10
11	11	11	11	11	11	11	11
12	12	12	12	12	12	12	12
13	13	13	13	13	13	13	13
14	14	14	14	14	14	14	14
15	15	15	15	15	15	15	15
16	16	16	16	16	16	16	16
17	17	17	17	17	17	17	17
18	18	18	18	18	18	18	18
19	19	19	19	19	19	19	19
20	20	20	20	20	20	20	20
21	21	21	21	21	21	21	21
22	22	22	22	22	22	22	22
23	23	23	23	23	23	23	23
24	24	24	24	24	24	24	24
25	25	25	25	25	25	25	25
26	26	26	26	26	26	26	26
27	27	27	27	27	27	27	27
28	28	28	28	28	28	28	28
29	29	29	29	29	29	29	29
30	30	30	30	30	30	30	30
31	31	31	31	31	31	31	31
32	32	32	32	32	32	32	32
33	33	33	33	33	33	33	33
34	34	34	34	34	34	34	34
35	35	35	35	35	35	35	35
36	36	36	36	36	36	36	36
37	37	37	37	37	37	37	37
38	38	38	38	38	38	38	38
39	39	39	39	39	39	39	39
40	40	40	40	40	40	40	40
41	41	41	41	41	41	41	41
42	42	42	42	42	42	42	42
43	43	43	43	43	43	43	43
44	44	44	44	44	44	44	44
45	45	45	45	45	45	45	45
46	46	46	46	46	46	46	46
47	47	47	47	47	47	47	47
48	48	48	48	48	48	48	48
49	49	49	49	49	49	49	49
50	50	50	50	50	50	50	50
51	51	51	51	51	51	51	51
52	52	52	52	52	52	52	52
53	53	53	53	53	53	53	53
54	54	54	54	54	54	54	54
55	55	55	55	55	55	55	55
56	56	56	56	56	56	56	56
57	57	57	57	57	57	57	57
58	58	58	58	58	58	58	58
59	59	59	59	59	59	59	59
60	60	60	60	60	60	60	60
61	61	61	61	61	61	61	61
62	62	62	62	62	62	62	62
63	63	63	63	63	63	63	63
64	64	64	64	64	64	64	64
65	65	65	65	65	65	65	65
66	66	66	66	66	66	66	66
67	67	67	67	67	67	67	67
68	68	68	68	68	68	68	68
69	69	69	69	69	69	69	69
70	70	70	70	70	70	70	70
71	71	71	71	71	71	71	71
72	72	72	72	72	72	72	72
73	73	73	73	73	73	73	73
74	74	74	74	74	74	74	74
75	75	75	75	75	75	75	75
76	76	76	76	76	76	76	76
77	77	77	77	77	77	77	77
78	78	78	78	78	78	78	78
79	79	79	79	79	79	79	79
80	80	80	80	80	80	80	80
81	81	81	81	81	81	81	81
82	82	82	82	82	82	82	82
83	83	83	83	83	83	83	83
84	84	84	84	84	84	84	84
85	85	85	85	85	85	85	85
86	86	86	86	86	86	86	86
87	87	87	87	87	87	87	87
88	88	88	88	88	88	88	88
89	89	89	89	89	89	89	89
90	90	90	90	90	90	90	90
91	91	91	91	91	91	91	91
92	92	92	92	92	92	92	92
93	93	93	93	93	93	93	93
94	94	94	94	94	94	94	94
95	95	95	95	95	95	95	95
96	96	96	96	96	96	96	96
97	97	97	97	97	97	97	97
98	98	98	98	98	98	98	98
99	99	99	99	99	99	99	99
100	100	100	100	100	100	100	100

TABLE 1.4

Observed Frequencies of the Interarrival Times
Between 12 P.M. and 1 P.M. of Each Day Considered

Length in 10ths of a Second = r	Number = n_r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
0-	4	17	8	10	15	5		15
1-	6	7	5	9	6	3		14
2-	5	12	12	8	12	4		13
3-	4	9	9	3	6	4		5
4-	1	4	5	3	7	2		5
5-	3	6	3	8	6	1		3
6-	3	7	8	2	3	2		4
7-	2	5	7	1	3	3		1
8-	3	1	7	2	0	2		1
9-	1	1	2	3	0	0		1
10-	1	2	0	4	3	2		4
11-	2	3	1	4	0	2		4
12-	0	1	1	0	0	0		1
13-	2	0	1	0	0	2		0
14-	1	0	1	1	1	0		0
15-	1	1	0	0	0	1		1
16-	0	0	0	0	0	0		1
17-	0	0	1	1	1	0		0
18-	1	0	0	1	1	0		0
19-	0	0	0	0	0	2		0
20-	0	0	0	0	2	0		0
21-	0	1	0	1	0	0		1
22-	0	0	0	0	0	1		0
23-	0	1	0	0	0	0		0
24-	1	0	0	1	0	0		1
25-	1	0	0	0	0	0		0
26-	0	0	0	0	0	1		0
27-	0	0	0	0	1	0		0
28-	0	0	0	0	0	0		0
$\geq$ 29-	1	0	0	0	1	2		0

43

78

71

62

68

39

75

TABLE 1.1

Observed Frequencies of the Interval Y between 11.11 and 11.12

Interval Y	Observed Frequencies					Total	
	11.11	11.12	11.13	11.14	11.15	11.16	11.17
11.11	1	1	1	1	1	5	11.11
11.12	1	1	1	1	1	5	11.12
11.13	1	1	1	1	1	5	11.13
11.14	1	1	1	1	1	5	11.14
11.15	1	1	1	1	1	5	11.15
11.16	1	1	1	1	1	5	11.16
11.17	1	1	1	1	1	5	11.17
11.18	1	1	1	1	1	5	11.18
11.19	1	1	1	1	1	5	11.19
11.20	1	1	1	1	1	5	11.20
11.21	1	1	1	1	1	5	11.21
11.22	1	1	1	1	1	5	11.22
11.23	1	1	1	1	1	5	11.23
11.24	1	1	1	1	1	5	11.24
11.25	1	1	1	1	1	5	11.25
11.26	1	1	1	1	1	5	11.26
11.27	1	1	1	1	1	5	11.27
11.28	1	1	1	1	1	5	11.28
11.29	1	1	1	1	1	5	11.29
11.30	1	1	1	1	1	5	11.30
11.31	1	1	1	1	1	5	11.31
11.32	1	1	1	1	1	5	11.32
11.33	1	1	1	1	1	5	11.33
11.34	1	1	1	1	1	5	11.34
11.35	1	1	1	1	1	5	11.35
11.36	1	1	1	1	1	5	11.36
11.37	1	1	1	1	1	5	11.37
11.38	1	1	1	1	1	5	11.38
11.39	1	1	1	1	1	5	11.39
11.40	1	1	1	1	1	5	11.40
11.41	1	1	1	1	1	5	11.41
11.42	1	1	1	1	1	5	11.42
11.43	1	1	1	1	1	5	11.43
11.44	1	1	1	1	1	5	11.44
11.45	1	1	1	1	1	5	11.45
11.46	1	1	1	1	1	5	11.46
11.47	1	1	1	1	1	5	11.47
11.48	1	1	1	1	1	5	11.48
11.49	1	1	1	1	1	5	11.49
11.50	1	1	1	1	1	5	11.50

TABLE 1.5

Observed Frequencies of the Interarrival Times
Between 1 P.M. and 2 P.M. of Each Day Considered

Length in 10ths of a Second = r	Number = n _r						
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th
0-	24	39	45	29	30	23	
1-	24	24	29	21	23	19	
2-	15	19	22	13	14	8	
3-	15	14	18	10	11	8	
4-	9	6	7	4	7	8	
5-	8	6	5	6	3	5	
6-	3	6	3	6	5	8	
7-	5	3	2	6	2	3	
8-	2	3	2	4	3	5	
9-	3	1	3	3	2	0	
10-	2	1	1	2	1	0	
11-	0	1	0	0	2	1	
12-	0	1	0	0	0	2	
13-	0	0	0	0	0	1	
14-	0	1	0	1	2	1	
15-	1	1	2	0	0	2	
16-	0	0	0	0	1	0	
17-	0	0	0	0	0	0	
18-	0	0	0	0	0	0	
19-	0	0	0	0	0	0	
20-	0	0	0	0	0	0	
21-	0	0	0	0	0	0	
22-	0	0	0	0	0	0	
23-	0	0	0	0	0	0	
24-	0	0	0	0	1	0	
25-	0	0	0	1	0	0	
26-	0	0	0	0	0	0	
27-	0	0	0	0	0	0	
28-	0	0	0	0	0	0	
≥ 29	0	0	0	0	0	0	

111

126

139

106

107

94

TABLE 1.6

Observed Frequencies of the Interarrival Times
Between 2 P.M. and 3 P.M. of Each Day Considered

Length in 10ths of a Second = r	Number = n _r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
0-	26	25	24	35	26	19	19	
1-	25	24	17	30	19	17	12	
2-	20	15	13	24	17	12	10	
3-	15	15	13	9	17	13	7	
4-	12	12	11	7	8	6	3	
5-	6	3	4	4	4	8	3	
6-	3	4	3	2	2	8	2	
7-	6	3	4	5	2	4	3	
8-	1	3	2	5	3	1	1	
9-	1	1	2	0	2	1	1	
10-	0	1	0	0	0	1	0	
11-	1	1	2	0	1	1	0	
12-	1	0	0	1	0	2	1	
13-	0	0	1	1	1	1	0	
14-	1	1	2	0	1	0	0	
15-	0	2	1	1	1	0	0	
16-	0	0	1	0	0	0	0	
17-	0	0	0	0	1	0	0	
18-	0	0	0	0	0	0	0	
19-	0	0	0	0	0	1	0	
20-	0	0	0	0	0	0	0	
21-	0	0	0	0	0	0	0	
22-	0	0	0	0	1	0	0	
23-	0	0	0	0	0	0	0	
24-	0	0	0	0	0	0	0	
25-	0	0	0	0	0	0	0	
26-	0	0	0	1	0	0	0	
27-	0	0	0	0	0	0	0	
28-	0	0	0	0	0	0	1	
> 29-	0	0	0	0	0	0	0	
	118	110	100	125	106	95	63	

Observed Frequencies of the Interval Lengths
Between 1 P.M. and 2 P.M. of 1920

Length in Tons of a Tramcar	0-1	1-2	2-3	3-4	4-5	5-6	6-7	7-8	8-9	9-10	10-11	11-12	12-13	13-14	14-15	15-16	16-17	17-18	18-19	19-20	20-21	21-22	22-23	23-24	24-25	25-26	26-27	27-28	28-29	29-30	30-31	31-32	32-33	33-34	34-35	35-36	36-37	37-38	38-39	39-40	40-41	41-42	42-43	43-44	44-45	45-46	46-47	47-48	48-49	49-50	50-51	51-52	52-53	53-54	54-55	55-56	56-57	57-58	58-59	59-60	60-61	61-62	62-63	63-64	64-65	65-66	66-67	67-68	68-69	69-70	70-71	71-72	72-73	73-74	74-75	75-76	76-77	77-78	78-79	79-80	80-81	81-82	82-83	83-84	84-85	85-86	86-87	87-88	88-89	89-90	90-91	91-92	92-93	93-94	94-95	95-96	96-97	97-98	98-99	99-100	100-101	101-102	102-103	103-104	104-105	105-106	106-107	107-108	108-109	109-110	110-111	111-112	112-113	113-114	114-115	115-116	116-117	117-118	118-119	119-120	120-121	121-122	122-123	123-124	124-125	125-126	126-127	127-128	128-129	129-130	130-131	131-132	132-133	133-134	134-135	135-136	136-137	137-138	138-139	139-140	140-141	141-142	142-143	143-144	144-145	145-146	146-147	147-148	148-149	149-150	150-151	151-152	152-153	153-154	154-155	155-156	156-157	157-158	158-159	159-160	160-161	161-162	162-163	163-164	164-165	165-166	166-167	167-168	168-169	169-170	170-171	171-172	172-173	173-174	174-175	175-176	176-177	177-178	178-179	179-180	180-181	181-182	182-183	183-184	184-185	185-186	186-187	187-188	188-189	189-190	190-191	191-192	192-193	193-194	194-195	195-196	196-197	197-198	198-199	199-200	200-201	201-202	202-203	203-204	204-205	205-206	206-207	207-208	208-209	209-210	210-211	211-212	212-213	213-214	214-215	215-216	216-217	217-218	218-219	219-220	220-221	221-222	222-223	223-224	224-225	225-226	226-227	227-228	228-229	229-230	230-231	231-232	232-233	233-234	234-235	235-236	236-237	237-238	238-239	239-240	240-241	241-242	242-243	243-244	244-245	245-246	246-247	247-248	248-249	249-250	250-251	251-252	252-253	253-254	254-255	255-256	256-257	257-258	258-259	259-260	260-261	261-262	262-263	263-264	264-265	265-266	266-267	267-268	268-269	269-270	270-271	271-272	272-273	273-274	274-275	275-276	276-277	277-278	278-279	279-280	280-281	281-282	282-283	283-284	284-285	285-286	286-287	287-288	288-289	289-290	290-291	291-292	292-293	293-294	294-295	295-296	296-297	297-298	298-299	299-300	300-301	301-302	302-303	303-304	304-305	305-306	306-307	307-308	308-309	309-310	310-311	311-312	312-313	313-314	314-315	315-316	316-317	317-318	318-319	319-320	320-321	321-322	322-323	323-324	324-325	325-326	326-327	327-328	328-329	329-330	330-331	331-332	332-333	333-334	334-335	335-336	336-337	337-338	338-339	339-340	340-341	341-342	342-343	343-344	344-345	345-346	346-347	347-348	348-349	349-350	350-351	351-352	352-353	353-354	354-355	355-356	356-357	357-358	358-359	359-360	360-361	361-362	362-363	363-364	364-365	365-366	366-367	367-368	368-369	369-370	370-371	371-372	372-373	373-374	374-375	375-376	376-377	377-378	378-379	379-380	380-381	381-382	382-383	383-384	384-385	385-386	386-387	387-388	388-389	389-390	390-391	391-392	392-393	393-394	394-395	395-396	396-397	397-398	398-399	399-400	400-401	401-402	402-403	403-404	404-405	405-406	406-407	407-408	408-409	409-410	410-411	411-412	412-413	413-414	414-415	415-416	416-417	417-418	418-419	419-420	420-421	421-422	422-423	423-424	424-425	425-426	426-427	427-428	428-429	429-430	430-431	431-432	432-433	433-434	434-435	435-436	436-437	437-438	438-439	439-440	440-441	441-442	442-443	443-444	444-445	445-446	446-447	447-448	448-449	449-450	450-451	451-452	452-453	453-454	454-455	455-456	456-457	457-458	458-459	459-460	460-461	461-462	462-463	463-464	464-465	465-466	466-467	467-468	468-469	469-470	470-471	471-472	472-473	473-474	474-475	475-476	476-477	477-478	478-479	479-480	480-481	481-482	482-483	483-484	484-485	485-486	486-487	487-488	488-489	489-490	490-491	491-492	492-493	493-494	494-495	495-496	496-497	497-498	498-499	499-500	500-501	501-502	502-503	503-504	504-505	505-506	506-507	507-508	508-509	509-510	510-511	511-512	512-513	513-514	514-515	515-516	516-517	517-518	518-519	519-520	520-521	521-522	522-523	523-524	524-525	525-526	526-527	527-528	528-529	529-530	530-531	531-532	532-533	533-534	534-535	535-536	536-537	537-538	538-539	539-540	540-541	541-542	542-543	543-544	544-545	545-546	546-547	547-548	548-549	549-550	550-551	551-552	552-553	553-554	554-555	555-556	556-557	557-558	558-559	559-560	560-561	561-562	562-563	563-564	564-565	565-566	566-567	567-568	568-569	569-570	570-571	571-572	572-573	573-574	574-575	575-576	576-577	577-578	578-579	579-580	580-581	581-582	582-583	583-584	584-585	585-586	586-587	587-588	588-589	589-590	590-591	591-592	592-593	593-594	594-595	595-596	596-597	597-598	598-599	599-600	600-601	601-602	602-603	603-604	604-605	605-606	606-607	607-608	608-609	609-610	610-611	611-612	612-613	613-614	614-615	615-616	616-617	617-618	618-619	619-620	620-621	621-622	622-623	623-624	624-625	625-626	626-627	627-628	628-629	629-630	630-631	631-632	632-633	633-634	634-635	635-636	636-637	637-638	638-639	639-640	640-641	641-642	642-643	643-644	644-645	645-646	646-647	647-648	648-649	649-650	650-651	651-652	652-653	653-654	654-655	655-656	656-657	657-658	658-659	659-660	660-661	661-662	662-663	663-664	664-665	665-666	666-667	667-668	668-669	669-670	670-671	671-672	672-673	673-674	674-675	675-676	676-677	677-678	678-679	679-680	680-681	681-682	682-683	683-684	684-685	685-686	686-687	687-688	688-689	689-690	690-691	691-692	692-693	693-694	694-695	695-696	696-697	697-698	698-699	699-700	700-701	701-702	702-703	703-704	704-705	705-706	706-707	707-708	708-709	709-710	710-711	711-712	712-713	713-714	714-715	715-716	716-717	717-718	718-719	719-720	720-721	721-722	722-723	723-724	724-725	725-726	726-727	727-728	728-729	729-730	730-731	731-732	732-733	733-734	734-735	735-736	736-737	737-738	738-739	739-740	740-741	741-742	742-743	743-744	744-745	745-746	746-747	747-748	748-749	749-750	750-751	751-752	752-753	753-754	754-755	755-756	756-757	757-758	758-759	759-760	760-761	761-762	762-763	763-764	764-765	765-766	766-767	767-768	768-769	769-770	770-771	771-772	772-773	773-774	774-775	775-776	776-777	777-778	778-779	779-780	780-781	781-782	782-783	783-784	784-785	785-786	786-787	787-788	788-789	789-790	790-791	791-792	792-793	793-794	794-795	795-796	796-797	797-798	798-799	799-800	800-801	801-802	802-803	803-804	804-805	805-806	806-807	807-808	808-809	809-810	810-811	811-812	812-813	813-814	814-815	815-816	816-817	817-818	818-819	819-820	820-821	821-822	822-823	823-824	824-825	825-826	826-827	827-828	828-829	829-830	830-831	831-832	832-833	833-834	834-835	835-836	836-837	837-838	838-839	839-840	840-841	841-842	842-843	843-844	844-845	845-846	846-847	847-848	848-849	849-850	850-851	851-852	852-853	853-854	854-855	855-856	856-857	857-858	858-859	859-860	860-861	861-862	862-863	863-864	864-865	865-866	866-867	867-868	868-869	869-870	870-871	871-872	872-873	873-874	874-875	875-876	876-877	877-878	878-879	879-880	880-881	881-882	882-883	883-884	884-885	885-886	886-887	887-888	888-889	889-890	890-891	891-892	892-893	893-894	894-895	895-896	896-897	897-898	898-899	899-900	900-901	901-902	902-903	903-904	904-905	905-906	906-907	907-908	908-909	909-910	910-911	911-912	912-913	913-914	914-915	915-916	916-917	917-918	918-919	919-920	920-921	921-922	922-923	923-924	924-925	925-926	926-927	927-928	928-929	929-930	930-931	931-932	932-933	933-934	934-935	935-936	936-937	937-938	938-939	939-940	940-941	941-942	942-943	943-944	944-945	945-946	946-947	947-948	948-949	949-950	950-951	951-952	952-953	953-954	954-955	955-956	956-957	957-958	958-959	959-960	960-961	961-962	962-963	963-964	964-965	965-966	966-967	967-968	968-969	969-970	970-971	971-972	972-973	973-974	974-975	975-976	976-977	977-978	978-979	979-980	980-981	981-982	982-983	983-984	984-985	985-986	986-987	987-988	988-989	989-990	990-991	991-992	992-993	993-994	994-995	995-996	996-997	997-998	998-999	999-1000	1000-1001	1001-1002	1002-1003	1003-1004	1004-1005	1005-1006	1006-1007	1007-1008	1008-1009	1009-1010	1010-1011	1011-1012	1012-1013	1013-1014	1014-1015	1015-1016	1016-1017	1017-1018	1018-1019	1019-1020	1020-1021	1021-1022	1022-1023	1023-1024	1024-1025	1025-1026	1026-1027	1027-1028	1028-1029	1029-1030	1030-1031	1031-10
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TABLE 1.7

Observed Frequencies of the Interarrival Times
Between 3 P.M. and 4 P.M. of Each Day Considered

Length in 10ths of a Second = r	Number = n _r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
0-	15	27	22	24	22	17	37	
1-	16	15	21	17	18	16	18	
2-	11	16	13	12	17	6	15	
3-	8	11	14	12	10	5	11	
4-	11	7	10	8	7	8	5	
5-	6	7	7	8	6	7	2	
6-	1	6	5	5	3	3	5	
7-	5	3	3	2	2	4	6	
8-	3	4	2	0	4	3	1	
9-	1	2	2	3	4	1	1	
10-	2	0	2	2	0	3	2	
11-	1	3	2	1	0	1	1	
12-	0	0	0	2	1	0	0	
13-	0	0	0	2	2	0	2	
14-	0	1	0	0	0	1	1	
15-	1	0	0	0	1	1	0	
16-	1	0	0	0	0	2	0	
17-	2	0	1	0	0	1	0	
18-	0	1	0	0	0	0	0	
19-	1	0	0	1	0	1	0	
20-	0	0	0	0	0	0	0	
21-	0	0	0	0	0	0	0	
22-	0	0	0	0	0	0	0	
23-	0	0	0	0	0	0	0	
24-	0	0	0	0	0	0	0	
25-	0	0	0	0	1	0	0	
26-	0	0	0	0	0	0	1	
27-	0	0	0	0	0	0	0	
28-	0	0	0	0	0	0	0	
≥ 29-	0	0	0	0	0	0	0	
	85	103	104	99	98	80	108	

TABLE 1.8

Observed Frequencies of the Interarrival Times
Between 4 P.M. and 5 P.M. of Each Day Considered

Length in 10ths of a Second = r	Number = n _r							
	Oct. 2nd	Oct. 3rd.	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
0-	21	21	16	23	15	14	26	
1-	16	13	15	24	7	13	21	
2-	8	11	9	9	15	7	12	
3-	10	8	4	9	8	7	12	
4-	7	5	4	9	8	7	4	
5-	1	4	4	5	11	4	5	
6-	4	3	2	5	0	4	5	
7-	3	4	3	2	4	3	2	
8-	5	3	3	2	2	1	3	
9-	3	1	2	2	1	3	0	
10-	1	1	0	2	2	1	1	
11-	2	1	3	1	1	0	1	
12-	0	1	1	0	0	2	0	
13-	1	1	1	1	0	1	1	
14-	0	1	1	1	1	0	2	
15-	0	1	1	1	1	1	0	
16-	0	0	0	0	0	1	1	
17-	2	1	0	0	0	0	0	
18-	0	0	0	0	1	1	0	
19-	1	0	0	1	0	0	1	
20-	0	0	0	0	0	0	1	
21-	1	0	1	0	0	0	0	
22-	0	1	0	1	0	1	0	
23-	0	0	0	0	0	0	0	
24-	0	0	0	0	0	0	0	
25-	0	0	0	0	1	0	0	
26-	0	0	0	0	0	0	0	
27-	0	1	0	0	1	1	0	
28-	0	0	0	0	0	0	0	
≥ 29	0	0	1	0	0	0	0	
	86	82	71	98	79	71	99	

TABLE 1.1

Interval Frequencies of the Interval Times
Between 1 P.M. and 2 P.M. of Each Day Considered

Number = n							Interval in seconds = r
Oct. 1st	Oct. 2nd	Oct. 3rd	Oct. 4th	Oct. 5th	Oct. 6th	Oct. 7th	Interval in seconds = r
1	1	1	1	1	1	1	0-
1	1	1	1	1	1	1	1-
1	1	1	1	1	1	1	2-
1	1	1	1	1	1	1	3-
1	1	1	1	1	1	1	4-
1	1	1	1	1	1	1	5-
1	1	1	1	1	1	1	6-
1	1	1	1	1	1	1	7-
1	1	1	1	1	1	1	8-
1	1	1	1	1	1	1	9-
1	1	1	1	1	1	1	10-
1	1	1	1	1	1	1	11-
1	1	1	1	1	1	1	12-
1	1	1	1	1	1	1	13-
1	1	1	1	1	1	1	14-
1	1	1	1	1	1	1	15-
1	1	1	1	1	1	1	16-
1	1	1	1	1	1	1	17-
1	1	1	1	1	1	1	18-
1	1	1	1	1	1	1	19-
1	1	1	1	1	1	1	20-
1	1	1	1	1	1	1	21-
1	1	1	1	1	1	1	22-
1	1	1	1	1	1	1	23-
1	1	1	1	1	1	1	24-
1	1	1	1	1	1	1	25-
1	1	1	1	1	1	1	26-
1	1	1	1	1	1	1	27-
1	1	1	1	1	1	1	28-
1	1	1	1	1	1	1	29-
1	1	1	1	1	1	1	30-
1	1	1	1	1	1	1	31-
1	1	1	1	1	1	1	32-
1	1	1	1	1	1	1	33-
1	1	1	1	1	1	1	34-
1	1	1	1	1	1	1	35-
1	1	1	1	1	1	1	36-
1	1	1	1	1	1	1	37-
1	1	1	1	1	1	1	38-
1	1	1	1	1	1	1	39-
1	1	1	1	1	1	1	40-
1	1	1	1	1	1	1	41-
1	1	1	1	1	1	1	42-
1	1	1	1	1	1	1	43-
1	1	1	1	1	1	1	44-
1	1	1	1	1	1	1	45-
1	1	1	1	1	1	1	46-
1	1	1	1	1	1	1	47-
1	1	1	1	1	1	1	48-
1	1	1	1	1	1	1	49-
1	1	1	1	1	1	1	50-
1	1	1	1	1	1	1	51-
1	1	1	1	1	1	1	52-
1	1	1	1	1	1	1	53-
1	1	1	1	1	1	1	54-
1	1	1	1	1	1	1	55-
1	1	1	1	1	1	1	56-
1	1	1	1	1	1	1	57-
1	1	1	1	1	1	1	58-
1	1	1	1	1	1	1	59-
1	1	1	1	1	1	1	60-
1	1	1	1	1	1	1	61-
1	1	1	1	1	1	1	62-
1	1	1	1	1	1	1	63-
1	1	1	1	1	1	1	64-
1	1	1	1	1	1	1	65-
1	1	1	1	1	1	1	66-
1	1	1	1	1	1	1	67-
1	1	1	1	1	1	1	68-
1	1	1	1	1	1	1	69-
1	1	1	1	1	1	1	70-
1	1	1	1	1	1	1	71-
1	1	1	1	1	1	1	72-
1	1	1	1	1	1	1	73-
1	1	1	1	1	1	1	74-
1	1	1	1	1	1	1	75-
1	1	1	1	1	1	1	76-
1	1	1	1	1	1	1	77-
1	1	1	1	1	1	1	78-
1	1	1	1	1	1	1	79-
1	1	1	1	1	1	1	80-
1	1	1	1	1	1	1	81-
1	1	1	1	1	1	1	82-
1	1	1	1	1	1	1	83-
1	1	1	1	1	1	1	84-
1	1	1	1	1	1	1	85-
1	1	1	1	1	1	1	86-
1	1	1	1	1	1	1	87-
1	1	1	1	1	1	1	88-
1	1	1	1	1	1	1	89-
1	1	1	1	1	1	1	90-
1	1	1	1	1	1	1	91-
1	1	1	1	1	1	1	92-
1	1	1	1	1	1	1	93-
1	1	1	1	1	1	1	94-
1	1	1	1	1	1	1	95-
1	1	1	1	1	1	1	96-
1	1	1	1	1	1	1	97-
1	1	1	1	1	1	1	98-
1	1	1	1	1	1	1	99-
1	1	1	1	1	1	1	100-

TABLE 1.9

Observed Frequencies of the Busy Times Between
9 A.M. and 10 A.M. of Each Day Considered

Length in 10ths of a Second = r	Number = n_r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
0-	5	2	4	2	2	3	2	5
1-	1	5	6	5	1	2	3	5
2-	5	8	5	12	11	7	5	11
3-	7	5	5	6	5	4	5	7
4-	6	3	8	3	9	6	7	12
5-	4	5	5	5	3	12	7	3
6-	2	5	3	9	4	6	8	4
7-	4	4	8	0	5	4	6	4
8-	5	2	1	4	6	2	2	4
9-	2	2	5	5	3	2	2	5
10-	3	3	0	5	3	1	4	1
11-	2	4	2	3	1	5	4	2
12-	4	3	4	1	2	0	6	3
13-	1	5	2	2	1	0	1	2
14-	1	4	3	2	1	2	3	2
15-	0	2	3	1	1	1	3	1
16-	0	1	2	3	3	1	2	3
17-	3	0	2	3	5	0	2	1
18-	1	2	2	2	2	4	0	1
19-	4	0	2	1	0	1	2	0
20-	1	1	0	1	3	2	0	0
21-	4	1	1	3	1	1	0	1
22-	0	0	0	1	3	0	0	0
23-	2	1	4	0	0	1	0	2
24-	1	2	2	0	0	2	3	3
25-	5	0	1	2	0	0	0	1
26-	0	3	1	0	1	2	1	0
27-	1	0	2	0	0	1	1	1
28-	0	1	0	1	0	2	1	0
29-	0	1	5	0	1	0	0	0

TABLE 1.9 continued

Length in 10ths of a Second = r	Number = n _r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
30-	0	1	0	0	0	0	1	0
31-	0	0	0	0	2	1	0	0
32-	2	0	0	2	1	0	0	2
33-	0	1	1	1	1	0	0	1
34-	1	0	1	1	0	1	0	1
35-	0	1	1	0	0	0	0	0
36-	0	0	0	0	1	2	0	1
37-	0	2	0	0	0	2	0	1
38-	0	0	0	0	1	0	0	0
39-	0	0	0	0	0	1	1	0
40-	0	1	0	0	0	0	1	0
41-	0	0	0	0	0	0	0	0
42-	0	1	0	0	0	0	0	0
43-	0	0	1	0	0	0	0	0
44-	2	1	0	0	0	1	0	2
45-	0	0	1	0	0	1	0	0
46-	0	0	0	0	1	0	2	0
47-	2	0	0	0	0	0	0	0
48-	0	1	0	0	1	0	2	0
49-	0	0	0	1	0	0	0	0
50-	0	1	0	1	0	0	0	0
51-	0	1	0	0	0	0	0	0
52-	0	0	0	1	0	0	0	1
53-	0	0	0	1	0	0	0	0
54-	0	0	0	0	0	0	0	0
55-	0	0	0	1	0	0	0	0
56-	0	0	1	0	0	0	0	0
57-	0	0	1	0	1	0	0	1
58-	0	0	0	0	0	1	0	0
<u>> 59-</u>	2	1	3	3	6	4	8	4
	82	88	98	94	93	88	95	98

TABLE 1.10

Observed Frequencies of the Busy Times Between
10 A.M. and 11 A.M. of Each Day Considered

Length in 10ths of a Second = r	Number = n_r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
0-	5	1	12	3	3	1	2	10
1-	4	7	6	2	4	5	2	4
2-	6	7	8	7	7	9	8	6
3-	11	9	5	6	9	6	8	11
4-	4	4	8	3	4	2	11	6
5-	6	8	8	0	9	5	7	2
6-	5	10	1	5	7	2	3	4
7-	3	3	3	3	2	2	7	2
8-	3	5	1	2	3	2	3	2
9-	2	3	5	5	2	0	3	9
10-	2	4	3	1	1	1	5	3
11-	3	3	4	1	2	0	2	3
12-	1	2	1	2	4	2	1	4
13-	2	3	3	1	4	4	2	3
14-	3	0	3	1	0	5	2	1
15-	1	2	0	3	2	1	2	2
16-	2	2	2	1	0	0	0	1
17-	1	2	4	3	1	1	1	1
18-	1	0	2	1	0	1	2	2
19-	1	2	0	0	0	0	4	2
20-	2	2	2	1	2	0	0	3
21-	1	4	2	0	2	1	1	3
22-	1	1	1	0	0	2	0	0
23-	0	1	1	0	2	0	0	0
24-	1	1	0	0	0	1	1	1
25-	1	2	0	1	0	0	2	3
26-	0	0	1	1	1	0	0	1
27-	1	1	3	0	1	1	0	2
28-	0	2	0	2	2	0	2	1
29-	2	0	1	0	0	1	0	0

TABLE 1.10 continued

Length in 10th of a Second = r	Number = n _r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
30-	0	1	2	0	1	0	1	0
31-	0	0	0	1	0	1	0	0
32-	1	0	1	0	1	0	0	0
33-	1	1	0	0	0	0	1	1
34-	0	1	0	1	3	0	2	0
35-	0	0	0	2	1	1	2	0
36-	0	0	0	0	0	2	1	1
37-	0	0	1	1	1	0	1	0
38-	1	0	1	1	0	0	0	0
39-	0	2	0	0	0	0	1	0
40-	0	0	0	0	0	1	0	1
41-	0	0	1	0	0	0	0	0
42-	0	0	0	0	0	1	0	0
43-	1	0	1	0	0	0	0	1
44-	0	0	0	0	0	0	0	0
45-	0	0	0	0	0	0	1	1
46-	0	0	0	0	0	0	0	0
47-	2	0	0	0	0	0	0	0
48-	0	0	0	0	0	1	0	0
49-	0	0	0	0	0	0	0	0
50-	0	0	0	1	1	0	0	0
51-	0	0	0	0	1	0	0	2
52-	0	0	0	2	0	0	0	0
53-	0	0	0	0	0	0	0	0
54-	1	0	0	0	0	1	1	0
55-	1	0	0	0	0	0	1	0
56-	0	0	0	0	0	1	0	0
57-	0	0	0	0	0	0	0	0
58-	0	0	0	0	0	0	0	0
≥ 59-	2	2	1	0	0	1	1	1
	85	98	98	64	83	65	94	100

TABLE 1.1
continued

TABLE 1.11
Observed Frequencies of the Busy Times Between
11 A.M. and 12 P.M. of Each Day Considered

Length in 10ths of a Second = r	Number = n _r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
0-	4	1	2	7	4	3	1	2
1-	7	3	4	5	4	4	1	5
2-	5	7	9	9	7	10	14	8
3-	10	9	12	12	8	6	5	8
4-	11	6	9	7	9	4	7	6
5-	4	1	5	6	9	3	3	8
6-	5	3	2	5	5	2	5	3
7-	3	1	5	1	1	4	2	6
8-	2	4	4	4	4	2	2	6
9-	2	2	4	5	1	5	3	1
10-	3	5	3	5	2	6	4	0
11-	2	4	1	2	2	2	4	3
12-	1	2	2	3	2	3	6	3
13-	2	2	4	1	5	3	1	4
14-	1	3	2	2	4	3	1	3
15-	2	1	1	3	1	3	1	4
16-	4	2	0	3	1	4	2	1
17-	0	2	1	1	1	1	2	2
18-	3	1	0	1	1	0	1	0
19-	3	4	0	0	1	1	2	2
20-	1	2	1	1	0	1	0	0
21-	2	0	0	3	1	0	1	0
22-	0	0	0	3	0	2	2	1
23-	0	1	2	1	0	0	0	0
24-	2	1	1	2	2	0	0	0
25-	0	1	2	1	0	1	0	0
26-	0	0	1	1	1	0	1	0
27-	0	1	0	1	3	3	2	0
28-	0	0	0	0	2	1	2	0
29-	1	0	0	0	1	0	1	1

TABLE 1.11 continued

Length in 10ths of a Second = r	Number = n _r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
30-	0	1	0	0	0	0	0	0
31-	0	0	0	0	0	0	0	0
32-	0	1	0	0	0	1	1	0
33-	0	0	0	0	1	1	0	0
34-	1	0	0	0	0	0	1	0
35-	0	0	0	0	0	1	0	0
36-	1	0	0	0	0	0	2	0
37-	0	0	0	0	0	0	0	0
38-	0	0	0	0	0	0	0	0
39-	2	0	0	0	1	0	1	0
40-	0	0	0	0	0	0	0	0
41-	1	0	1	0	0	0	0	0
42-	1	0	0	0	1	1	0	0
43-	0	0	0	0	1	1	0	0
44-	0	0	1	1	0	0	0	0
45-	0	0	0	2	0	0	0	0
46-	0	0	0	0	0	0	0	0
47-	0	0	0	0	0	0	0	0
48-	0	0	0	0	0	0	0	0
49-	0	0	0	0	0	0	1	0
50-	0	0	0	0	0	0	0	0
51-	0	0	0	0	0	0	1	0
52-	2	0	0	0	2	1	0	0
53-	0	0	0	0	0	0	0	0
54-	0	0	0	0	0	1	0	0
55-	0	0	0	0	0	0	0	0
56-	0	0	0	0	1	0	0	0
57-	0	0	0	0	0	0	0	0
58-	1	0	0	0	0	1	0	0
<u>> 59</u>	0	3	2	7	3	3	2	2
	89	74	81	104	92	88	85	79

TABLE 1.12

Observed Frequencies of the Busy Times Between
12 P.M. and 1 P.M. of Each Day Considered

Length in 10ths of a Second = r	Number = n_r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
0-	0	8	3	5	1	0		0
1-	1	2	2	3	3	0		5
2-	4	11	8	4	0	1		11
3-	7	13	4	4	10	3		8
4-	3	7	3	9	7	7		6
5-	2	5	4	5	7	3		5
6-	2	6	4	4	9	5		5
7-	2	3	6	1	3	0		2
8-	6	3	3	5	5	0		6
9-	1	1	3	3	0	0		1
10-	1	1	3	2	2	0		3
11-	3	0	2	3	1	3		0
12-	0	1	4	1	1	1		3
13-	1	1	1	2	1	0		2
14-	1	2	0	0	2	1		3
15-	0	1	0	1	0	1		2
16-	1	1	1	1	4	0		2
17-	2	1	4	1	3	1		1
18-	0	1	3	1	0	0		1
19-	0	1	2	1	1	0		1
20-	0	0	1	0	1	0		2
21-	2	1	1	0	1	0		1
22-	0	1	1	1	0	1		0
23-	0	0	1	0	1	1		1
24-	2	0	1	1	0	0		0
25-	0	0	0	1	0	0		0
26-	0	0	0	0	1	0		0
27-	0	0	1	0	0	0		1
28-	0	0	1	0	0	0		1
29-	0	0	1	0	0	2		0

TABLE I-1

Summary of the results of the investigation of the ...
IX B.M. and I ... of the ...

Location		10th of a Second = r		Number		n	
Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 14th	Oct. 15th	Oct. 16th	Oct. 17th	Oct. 18th
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9
10	10	10	10	10	10	10	10
11	11	11	11	11	11	11	11
12	12	12	12	12	12	12	12
13	13	13	13	13	13	13	13
14	14	14	14	14	14	14	14
15	15	15	15	15	15	15	15
16	16	16	16	16	16	16	16
17	17	17	17	17	17	17	17
18	18	18	18	18	18	18	18
19	19	19	19	19	19	19	19
20	20	20	20	20	20	20	20
21	21	21	21	21	21	21	21
22	22	22	22	22	22	22	22
23	23	23	23	23	23	23	23
24	24	24	24	24	24	24	24
25	25	25	25	25	25	25	25
26	26	26	26	26	26	26	26
27	27	27	27	27	27	27	27
28	28	28	28	28	28	28	28
29	29	29	29	29	29	29	29
30	30	30	30	30	30	30	30
31	31	31	31	31	31	31	31
32	32	32	32	32	32	32	32
33	33	33	33	33	33	33	33
34	34	34	34	34	34	34	34
35	35	35	35	35	35	35	35
36	36	36	36	36	36	36	36
37	37	37	37	37	37	37	37
38	38	38	38	38	38	38	38
39	39	39	39	39	39	39	39
40	40	40	40	40	40	40	40
41	41	41	41	41	41	41	41
42	42	42	42	42	42	42	42
43	43	43	43	43	43	43	43
44	44	44	44	44	44	44	44
45	45	45	45	45	45	45	45
46	46	46	46	46	46	46	46
47	47	47	47	47	47	47	47
48	48	48	48	48	48	48	48
49	49	49	49	49	49	49	49
50	50	50	50	50	50	50	50
51	51	51	51	51	51	51	51
52	52	52	52	52	52	52	52
53	53	53	53	53	53	53	53
54	54	54	54	54	54	54	54
55	55	55	55	55	55	55	55
56	56	56	56	56	56	56	56
57	57	57	57	57	57	57	57
58	58	58	58	58	58	58	58
59	59	59	59	59	59	59	59
60	60	60	60	60	60	60	60
61	61	61	61	61	61	61	61
62	62	62	62	62	62	62	62
63	63	63	63	63	63	63	63
64	64	64	64	64	64	64	64
65	65	65	65	65	65	65	65
66	66	66	66	66	66	66	66
67	67	67	67	67	67	67	67
68	68	68	68	68	68	68	68
69	69	69	69	69	69	69	69
70	70	70	70	70	70	70	70
71	71	71	71	71	71	71	71
72	72	72	72	72	72	72	72
73	73	73	73	73	73	73	73
74	74	74	74	74	74	74	74
75	75	75	75	75	75	75	75
76	76	76	76	76	76	76	76
77	77	77	77	77	77	77	77
78	78	78	78	78	78	78	78
79	79	79	79	79	79	79	79
80	80	80	80	80	80	80	80
81	81	81	81	81	81	81	81
82	82	82	82	82	82	82	82
83	83	83	83	83	83	83	83
84	84	84	84	84	84	84	84
85	85	85	85	85	85	85	85
86	86	86	86	86	86	86	86
87	87	87	87	87	87	87	87
88	88	88	88	88	88	88	88
89	89	89	89	89	89	89	89
90	90	90	90	90	90	90	90
91	91	91	91	91	91	91	91
92	92	92	92	92	92	92	92
93	93	93	93	93	93	93	93
94	94	94	94	94	94	94	94
95	95	95	95	95	95	95	95
96	96	96	96	96	96	96	96
97	97	97	97	97	97	97	97
98	98	98	98	98	98	98	98
99	99	99	99	99	99	99	99
100	100	100	100	100	100	100	100

TABLE 1.12 continued

Length in 10ths of a Second = r	Number = n _r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
30-	0	0	0	0	1	0		1
31-	0	0	0	0	0	1		0
32-	0	0	1	0	0	0		0
33-	0	0	0	1	0	0		0
34-	0	0	0	0	0	1		0
35-	0	0	0	1	0	0		0
36-	0	1	1	0	0	1		0
37-	0	0	0	0	1	1		0
38-	0	0	0	0	0	0		0
39-	0	0	0	0	0	0		0
40-	0	0	0	0	0	0		1
41-	0	2	0	0	0	0		0
42-	0	0	0	0	0	0		0
43-	0	0	0	1	0	0		0
44-	0	0	0	0	0	1		0
45-	0	0	0	0	0	0		0
46-	0	0	0	0	0	1		0
47-	0	0	0	0	0	0		0
48-	0	0	0	0	0	0		0
49-	0	0	0	0	0	0		0
50-	0	1	1	0	0	1		0
51-	0	0	0	0	0	0		0
52-	0	0	0	0	0	0		0
53-	0	1	0	0	0	0		0
54-	0	0	0	0	0	0		0
55-	0	0	0	0	0	1		0
56-	0	0	0	0	0	0		0
57-	0	0	0	0	0	0		0
58-	0	0	0	0	0	0		0
≥ 59-	3	2	0	0	2	1		0
	43	78	71	62	68	39		75

TABLE 1. 12 continued

Length in feet of a	Second = r	Number = n					
		Oct. 3rd	Oct. 3rd	Oct. 6th	Oct. 9th	Oct. 12th	Oct. 14th
30-	0	0	0	0	0	0	0
31-	0	0	0	0	0	0	0
32-	0	0	0	0	0	0	0
33-	0	0	0	0	0	0	0
34-	0	0	0	0	0	0	0
35-	0	0	0	0	0	0	0
36-	0	0	0	0	0	0	0
37-	0	0	0	0	0	0	0
38-	0	0	0	0	0	0	0
39-	0	0	0	0	0	0	0
40-	0	0	0	0	0	0	0
41-	0	0	0	0	0	0	0
42-	0	0	0	0	0	0	0
43-	0	0	0	0	0	0	0
44-	0	0	0	0	0	0	0
45-	0	0	0	0	0	0	0
46-	0	0	0	0	0	0	0
47-	0	0	0	0	0	0	0
48-	0	0	0	0	0	0	0
49-	0	0	0	0	0	0	0
50-	0	0	0	0	0	0	0
51-	0	0	0	0	0	0	0
52-	0	0	0	0	0	0	0
53-	0	0	0	0	0	0	0
54-	0	0	0	0	0	0	0
55-	0	0	0	0	0	0	0
56-	0	0	0	0	0	0	0
57-	0	0	0	0	0	0	0
58-	0	0	0	0	0	0	0
59-	0	0	0	0	0	0	0
60-	0	0	0	0	0	0	0

TABLE 1.13

Observed Frequencies of the Busy Times Between
1 P.M. and 2 P.M. of Each Day Considered

Length in 10ths of a Second = r	Number = n _r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
0-	6	6	21	3	7	4		
1-	6	11	6	9	6	6		
2-	13	12	13	11	10	9		
3-	10	8	13	13	9	10		
4-	7	11	9	8	9	9		
5-	8	9	7	7	8	7		
6-	7	5	6	7	2	5		
7-	2	7	5	4	3	6		
8-	2	2	6	2	5	1		
9-	5	5	4	2	2	2		
10-	2	2	6	0	6	3		
11-	0	3	3	5	3	2		
12-	3	5	2	2	1	2		
13-	2	3	3	3	7	2		
14-	3	4	4	4	1	0		
15-	4	1	5	4	0	0		
16-	1	4	2	0	0	3		
17-	2	4	1	1	4	4		
18-	2	2	2	2	3	1		
19-	1	1	0	0	2	2		
20-	5	2	0	4	1	0		
21-	1	3	1	1	1	2		
22-	0	3	1	0	0	1		
23-	0	1	1	1	3	0		
24-	2	1	1	2	1	1		
25-	0	2	1	2	0	1		
26-	1	0	1	1	3	2		
27-	4	1	0	1	1	0		
28-	0	0	1	1	1	1		
29-	0	0	1	1	0	0		

TABLE 1.13 continued

Length in 10ths of a Second = r	Number = n _r						
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th
30-	0	0	0	0	0	0	
31-	1	0	0	1	2	0	
32-	0	0	1	1	0	0	
33-	1	0	0	0	0	1	
34-	0	2	0	0	0	0	
35-	1	0	0	1	1	0	
36-	2	0	0	0	0	1	
37-	1	0	0	1	1	0	
38-	1	0	0	0	0	0	
39-	0	1	0	1	0	1	
40-	0	0	0	0	0	0	
41-	0	0	0	0	0	1	
42-	1	0	1	0	0	2	
43-	0	0	0	0	0	0	
44-	0	0	2	0	1	0	
45-	0	0	0	0	0	0	
46-	0	1	0	0	0	0	
47-	1	1	1	0	0	0	
48-	0	0	0	0	0	0	
49-	0	0	0	0	0	0	
50-	0	0	1	0	0	0	
51-	0	0	0	0	0	0	
52-	0	0	1	0	0	0	
53-	0	0	0	0	0	0	
54-	0	1	0	0	0	0	
55-	0	0	0	0	0	0	
56-	0	0	0	0	0	0	
57-	0	0	0	0	0	0	
58-	0	0	0	0	0	0	
> 59-	3	2	6	0	3	2	

111

126

139

106

107

94

TABLE 1.14

Observed Frequencies of the Busy Times Between
2 P.M. and 3 P.M. of Each Day Considered

Length in 10ths of a Second = r	Number = n_r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
0-	10	5	12	4	1	11	5	
1-	12	6	12	10	9	9	3	
2-	15	12	13	11	9	8	8	
3-	7	8	9	11	6	3	8	
4-	12	7	7	9	8	8	7	
5-	6	7	8	7	5	5	5	
6-	4	8	2	7	2	6	3	
7-	2	5	4	5	5	3	4	
8-	2	5	4	3	6	3	1	
9-	3	2	2	3	5	3	2	
10-	1	2	3	1	3	2	2	
11-	4	2	2	2	2	2	2	
12-	2	3	3	2	4	4	1	
13-	1	1	2	3	3	2	3	
14-	1	0	1	5	1	2	0	
15-	1	2	1	5	1	1	0	
16-	3	2	0	2	2	0	0	
17-	2	4	3	2	3	3	0	
18-	0	2	1	2	1	0	1	
19-	0	1	1	4	1	0	1	
20-	2	0	1	2	3	1	1	
21-	0	3	0	3	1	1	0	
22-	1	1	0	1	0	0	0	
23-	1	0	0	3	0	1	1	
24-	2	1	0	0	3	1	0	
25-	2	2	0	1	0	0	0	
26-	2	0	2	1	1	1	0	
27-	0	1	0	4	0	1	0	
28-	0	1	1	1	1	0	0	
29-	1	1	0	0	2	1	0	

TABLE 1.14 continued

Length in 10ths of a Second = r	Number = n _r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
30-	4	1	0	0	0	1	0	
31-	2	0	0	0	1	1	0	
32-	1	2	1	1	2	1	0	
33-	1	2	0	3	1	0	1	
34-	0	2	0	0	0	1	0	
35-	0	0	0	0	0	2	1	
36-	0	1	1	0	1	0	0	
37-	2	0	0	1	0	2	0	
38-	1	1	0	0	0	0	0	
39-	0	0	0	0	1	1	0	
40-	0	0	0	1	2	0	0	
41-	0	1	1	0	1	0	0	
42-	0	1	1	0	0	0	0	
43-	0	0	1	0	0	0	0	
44-	0	0	0	0	1	0	0	
45-	0	0	0	1	1	0	0	
46-	0	1	0	0	0	0	0	
47-	0	1	0	0	0	0	1	
48-	0	1	0	0	0	0	0	
49-	1	0	1	0	2	0	0	
50-	0	0	0	0	0	0	1	
51-	0	0	0	0	1	0	0	
52-	0	0	0	0	0	0	0	
53-	0	0	0	1	0	0	0	
54-	0	0	0	0	0	0	0	
55-	0	0	0	0	0	0	0	
56-	0	0	0	0	1	0	0	
57-	0	0	0	0	0	0	1	
58-	1	0	0	0	0	0	0	
≥ 59	6	2	0	3	3	4	0	

118

110

100

125

106

95

63

TABLE 1.15

Observed Frequencies of the Busy Times Between
3 P.M. and 4 P.M. of Each Day Considered

Length in 10ths of a Second = r	Number = n_r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
0-	2	10	7	3	8	16	12	
1-	4	4	4	7	5	2	5	
2-	3	6	15	12	7	8	7	
3-	9	11	4	4	7	1	8	
4-	8	15	10	9	10	7	9	
5-	6	4	5	6	3	2	4	
6-	6	4	4	7	3	2	5	
7-	7	8	3	6	3	4	2	
8-	3	3	6	5	3	7	3	
9-	6	4	6	3	5	2	5	
10-	4	3	5	1	5	2	7	
11-	1	3	4	2	5	2	3	
12-	3	3	2	1	4	1	2	
13-	1	1	1	1	2	0	5	
14-	1	4	2	1	0	1	2	
15-	1	3	1	2	2	1	1	
16-	0	3	1	3	1	1	3	
17-	1	1	2	1	1	0	1	
18-	1	0	2	1	0	0	1	
19-	3	2	0	2	3	2	3	
20-	1	1	2	3	0	0	1	
21-	0	1	0	1	2	0	3	
22-	0	0	2	2	2	2	2	
23-	0	0	1	1	3	0	2	
24-	2	2	2	0	2	2	1	
25-	0	1	0	0	0	1	0	
26-	0	1	0	0	0	2	3	
27-	1	0	0	0	1	0	0	
28-	0	1	1	1	1	0	1	
29-	1	0	1	1	0	2	0	

TABLE 1.15 continued

Length in 10ths of a Second = r	Number = n _r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
30-	1	0	0	0	2	1	0	
31-	1	0	0	1	1	0	0	
32-	0	0	0	2	1	0	0	
33-	0	0	0	1	0	0	1	
34-	0	0	1	2	0	2	1	
35-	1	0	1	0	0	0	1	
36-	0	0	0	1	0	0	0	
37-	0	1	1	0	0	0	0	
38-	1	1	1	1	0	0	0	
39-	0	0	1	0	1	0	1	
40-	0	0	0	0	0	0	0	
41-	0	0	2	0	0	0	0	
42-	0	0	0	1	0	0	0	
43-	0	0	0	0	0	0	0	
44-	0	0	0	0	1	0	0	
45-	0	0	1	0	0	0	0	
46-	1	1	0	0	0	0	0	
47-	0	0	0	0	0	0	0	
48-	0	0	0	0	0	1	0	
49-	0	0	0	0	0	0	0	
50-	0	0	0	0	1	0	1	
51-	0	0	0	0	0	0	0	
52-	1	0	0	0	0	0	0	
53-	0	0	0	0	0	0	0	
54-	0	0	0	0	0	0	0	
55-	1	0	0	0	0	0	0	
56-	0	0	0	0	0	0	0	
57-	0	0	0	1	0	0	0	
58-	0	0	0	0	0	0	0	
≥ 59	3	1	4	2	2	6	2	

85

103

104

99

98

80

108

TABLE 1.16

Observed Frequencies of the Busy Times Between
4 P.M. and 5 P.M. of Each Day Considered

Length in 10ths of a Second = r	Number = n _r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
0-	5	2	6	0	8	4	8	
1-	9	2	2	7	4	1	4	
2-	13	13	9	9	8	6	7	
3-	10	7	8	11	6	6	12	
4-	5	11	5	9	7	8	8	
5-	2	5	4	6	7	5	10	
6-	5	4	4	5	3	3	8	
7-	4	3	3	4	2	1	8	
8-	4	4	2	6	0	6	2	
9-	3	2	3	7	3	2	2	
10-	2	5	2	3	1	1	6	
11-	3	1	3	3	2	1	1	
12-	1	1	1	1	1	2	4	
13-	1	2	1	2	4	6	3	
14-	1	2	3	1	1	2	0	
15-	1	4	3	1	2	3	2	
16-	2	0	3	2	2	1	2	
17-	1	1	3	1	2	3	0	
18-	3	0	2	2	1	0	1	
19-	0	2	0	2	1	0	0	
20-	0	0	1	1	0	1	0	
21-	0	0	1	2	1	0	0	
22-	0	1	0	2	2	1	0	
23-	0	0	0	2	2	0	1	
24-	1	0	0	1	1	0	1	
25-	0	0	0	0	0	0	0	
26-	0	2	0	1	0	0	0	
27-	0	1	0	3	0	0	1	
28-	1	1	0	1	1	0	1	
29-	1	0	0	1	1	0	1	

TABLE 1.13

Observed frequencies of the busy times in two
P.M. and 7 P.M. of 1 day. Considered

Length in Tolls of a Second = r	Number of						Total
	2nd	3rd	4th	5th	6th	7th	
0	0	1	0	0	0	0	1
1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0
28	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0

TABLE 1.16 continued

Length in 10ths of a Second = r	Number = n _r							
	Oct. 2nd	Oct. 3rd	Oct. 6th	Oct. 7th	Oct. 8th	Oct. 9th	Oct. 10th	Oct. 14th
30-	0	0	0	0	0	0	0	
31-	2	2	0	0	1	1	0	
32-	0	1	1	0	0	0	0	
33-	0	0	0	0	0	1	0	
34-	0	1	0	0	0	0	1	
35-	0	1	0	0	0	0	0	
36-	0	0	0	1	0	0	0	
37-	1	0	0	0	0	0	0	
38-	0	0	1	1	0	1	0	
39-	1	1	0	0	0	1	0	
40-	0	0	0	0	1	1	0	
41-	0	0	0	0	0	0	0	
42-	0	0	0	0	0	0	0	
43-	1	0	1	0	2	0	0	
44-	0	0	0	0	0	0	0	
45-	0	0	0	0	0	0	0	
46-	0	0	0	0	0	0	0	
47-	0	0	0	0	0	0	1	
48-	0	0	0	0	1	0	0	
49-	0	0	0	0	0	1	1	
50-	0	0	0	0	1	0	0	
51-	0	0	0	0	0	0	0	
52-	0	0	0	0	0	0	0	
53-	1	0	0	0	0	1	0	
54-	0	0	0	0	0	0	0	
55-	1	0	0	0	0	0	0	
56-	0	0	0	0	0	0	0	
57-	0	0	0	0	0	0	0	
58-	0	0	0	0	0	0	0	
> 59	1	0	1	0	0	2	2	

86

82

71

98

79

71

99

TABLE 1.17

Maximum Likelihood Estimates of λ for
Each Hour of Each Day

	9-10	10-11	11-12	12-1	1-2	2-3	3-4	4-5
Oct. 2nd	.023	.023	.022	.013	.031	.033	.023	.023
Oct. 3rd	.024	.027	.020	.022	.034	.031	.028	.023
Oct. 6th	.026	.027	.022	.020	.038	.027	.029	.019
Oct. 7th	.026	.027	.029	.017	.028	.016	.028	.026
Oct. 8th	.025	.023	.025	.019	.029	.029	.026	.021
Oct. 9th	.024	.018	.025	.011	.026	.026	.022	.020
Oct. 10th	.025	.026	.023	—	—	.031	.031	.027
Oct. 14th	.025	.028	.021	.022	—	—	—	—

TABLE 1.18

Maximum Likelihood Estimates of μ for
Each Hour of Each Day

	9-10	10-11	11-12	12-1	1-2	2-3	3-4	4-5
Oct. 2nd	.0073	.0077	.0088	.0079	.0072	.0064	.0084	.0088
Oct. 3rd	.0070	.0091	.0085	.0092	.0084	.0069	.011	.011
Oct. 6th	.0064	.0084	.0098	.0095	.0074	.011	.0061	.010
Oct. 7th	.0061	.0075	.0066	.0013	.011	.0072	.0080	.011
Oct. 8th	.0058	.0092	.0073	.012	.0075	.0056	.0077	.0088
Oct. 9th	.0063	.0065	.0060	.0070	.0091	.0060	.0042	.0072
Oct. 10th	.0058	.0085	.0063	—	—	.011	.0082	.011
Oct. 14th	.0067	.0077	.011	.012	—	—	—	—

TABLE 1.19

Estimates of θ_1 for Each Hour of Each Day

	9-10	10-11	11-12	12-1	1-2	2-3	3-4	4-5
Oct. 2nd	.80	.79	.80	.87	.74	.72	.79	.79
Oct. 3rd	.78	.76	.82	.81	.71	.74	.76	.80
Oct. 6th	.77	.76	.80	.82	.68	.76	.75	.82
Oct. 7th	.77	.76	.75	.84	.75	.85	.76	.77
Oct. 8th	.78	.79	.78	.82	.75	.75	.77	.81
Oct. 9th	.79	.84	.78	.90	.77	.77	.80	.82
Oct. 10th	.78	.77	.79	—	—	.73	.74	.76
Oct. 14th	.78	.75	.81	.81	—	—	—	—

TABLE 1.20

Estimates of θ_2 for Each Hour of Each Day

	9-10	10-11	11-12	12-1	1-2	2-3	3-4	4-5
Oct. 2nd	.93	.93	.92	.92	.93	.94	.92	.92
Oct. 3rd	.93	.91	.92	.91	.92	.93	.89	.90
Oct. 6th	.94	.92	.91	.91	.93	.90	.94	.90
Oct. 7th	.94	.93	.94	.88	.90	.93	.92	.89
Oct. 8th	.94	.91	.93	.89	.93	.95	.93	.92
Oct. 9th	.94	.94	.94	.93	.91	.94	.96	.93
Oct. 10th	.94	.92	.94	—	—	.90	.92	.89
Oct. 14th	.94	.93	.90	.88	—	—	—	—

APPENDIX II: Tables of total
hourly results

TABLE 2.1

Total Observed Frequencies of the Interarrival Times
for Each One Hour Period

Length in 10ths of a Second = r	Number = n _r							
	9-10	10-11	11-12	12-1	1-2	2-3	3-4	4-5
0-	168	145	144	74	190	174	164	135
1-	121	123	124	50	140	144	121	109
2-	94	97	104	66	91	111	90	71
3-	85	73	70	40	76	89	71	58
4-	63	60	48	27	41	59	56	44
5-	44	39	34	30	33	32	43	34
6-	33	33	40	29	31	24	28	23
7-	35	27	28	22	21	27	25	22
8-	22	21	25	16	19	16	17	19
9-	12	15	14	8	12	8	14	12
10-	7	13	14	16	7	2	11	8
11-	6	10	9	16	4	6	9	9
12-	12	5	7	3	3	5	3	4
13-	6	3	4	5	1	4	6	6
14-	7	6	6	4	5	5	3	6
15-	4	3	5	4	6	5	3	5
16-	6	4	2	1	1	1	3	2
17-	2	2	0	3	0	1	4	3
18-	2	2	2	3	0	0	1	2
19-	2	2	2	2	0	1	3	3
20-	2	1	1	2	0	0	0	1
21-	1	0	1	3	0	0	0	2
22-	1	0	1	1	0	1	0	3
23-	0	0	1	1	0	0	0	0
24-	1	0	0	3	1	0	0	0
25-	0	0	1	1	1	0	1	1
26-	0	0	0	1	0	1	1	0
27-	0	2	0	1	0	0	0	3
28-	0	0	0	0	0	1	0	0
> 29-	0	1	5	4	0	0	0	1
	736	687	692	436	683	717	677	586

100 300 1.5 1

TABLE 2.2

Total Observed Frequencies of the Busy Times for Each
One Hour Period

Length in 10ths of a Second = r	Number = n_r							
	9-10	10-11	11-12	12-1	1-2	2-3	3-4	4-5
0-	28	39	25	16	47	48	60	34
1-	28	34	33	16	44	61	31	29
2-	64	58	69	39	68	76	58	65
3-	44	65	70	49	63	52	44	60
4-	54	42	59	42	53	58	68	53
5-	44	45	38	31	46	43	30	39
6-	41	37	30	35	32	32	31	32
7-	35	25	23	17	27	28	33	25
8-	26	21	28	28	18	24	30	24
9-	26	29	23	9	20	20	29	22
10-	20	20	28	12	19	14	27	20
11-	23	18	20	12	16	16	20	14
12-	23	17	22	11	15	19	16	11
13-	14	22	21	8	20	15	11	19
14-	18	15	19	9	16	10	11	10
15-	12	13	16	5	14	11	11	15
16-	15	8	17	10	10	9	12	11
17-	16	14	10	13	16	17	7	11
18-	14	8	7	6	12	7	5	9
19-	10	9	13	6	6	8	15	5
20-	8	12	6	4	12	10	8	3
21-	12	14	7	6	9	8	7	4
22-	4	5	8	4	5	3	10	6
23-	10	4	4	4	6	6	7	5
24-	13	5	8	4	8	7	11	4
25-	9	9	5	1	6	5	2	0
26-	8	4	4	1	8	7	6	3
27-	6	9	10	2	7	6	2	5
28-	3	9	5	2	4	4	5	5
29-	7	4	4	3	2	5	5	4

TABLE 2.2 continued

Length in 10ths of a Second = r	Number = n _r							
	9-10	10-11	11-12	12-1	1-2	2-3	3-4	4-5
30-	2	5	1	2	0	6	4	0
31-	3	2	0	1	4	4	3	6
32-	7	3	3	1	2	8	3	2
33-	5	4	2	1	2	8	2	1
34-	5	7	2	1	2	3	6	2
35-	2	6	1	1	3	3	3	1
36-	4	4	3	3	3	3	1	1
37-	5	4	0	2	3	5	2	1
38-	1	3	0	0	1	2	4	3
39-	2	3	4	0	3	2	3	3
40-	2	2	0	1	0	3	0	2
41-	0	1	2	2	1	3	2	0
42-	1	1	3	0	4	2	1	0
43-	1	3	2	1	0	1	0	4
44-	6	0	2	1	3	1	1	0
45-	2	2	2	0	0	2	1	0
46-	3	0	0	1	1	1	2	0
47-	2	2	0	0	3	2	0	1
48-	4	1	0	0	0	1	1	1
49-	1	0	1	0	0	4	0	2
50-	2	2	0	3	1	1	2	0
51-	1	2	1	0	0	1	0	0
52-	2	2	5	0	1	0	1	0
53-	1	0	0	1	0	1	0	2
54-	0	3	1	0	1	0	0	0
55-	1	2	0	1	0	0	1	1
56-	1	1	1	0	0	1	0	0
57-	3	0	0	0	0	1	1	0
58-	1	0	2	0	0	1	0	0
<u>≥</u> 59-	31	8	22	8	16	18	21	6

736

687

692

436

683

717

677

586

TABLE 2.3

Maximum Likelihood Estimates of λ and μ for
Each One Hour Period

TIME	9-10	10-11	11-12	12-1	1-2	2-3	3-4	4-5
Number of Calls = No	736	687	692	436	683	717	677	586
Amount of Time Analysed	8 hrs	7 hrs 39 mins	8 hrs	7 hrs	6 hrs	6 hrs 30 mins	7 hrs	7 hrs
$\hat{\lambda}$	.026	.025	.026	.018	.032	.030	.027	.023
$\hat{\mu}$	.0063	.0093	.0077	.0093	.0082	.0071	.0074	.0099

TABLE 2.4

Estimates of θ_1 and θ_2 for Each One Hour Period

TIME	9-10	10-11	11-12	12-1	1-2	2-3	3-4	4-5
$\hat{\theta}_1$	.77	.78	.77	.84	.73	.74	.77	.80
$\hat{\theta}_2$	.94	.91	.93	.91	.92	.93	.93	.91

TABLE 2.5
The Results of the χ^2 Tests

TIME	9-10	10-11	11-12	12-1	1-2	2-3	3-4	4-5
χ^2_{10} for T_1 Distributions	11	1.9	11	10	8.1	17	2.5	12
χ^2_{22} for Y Distributions	35	55	58	45	42	53	53	37

APPENDIX III: Graphical representations of
total hourly results

FIGURE 3.1

Interarrival Times - 9 A.M. to 10 A.M.

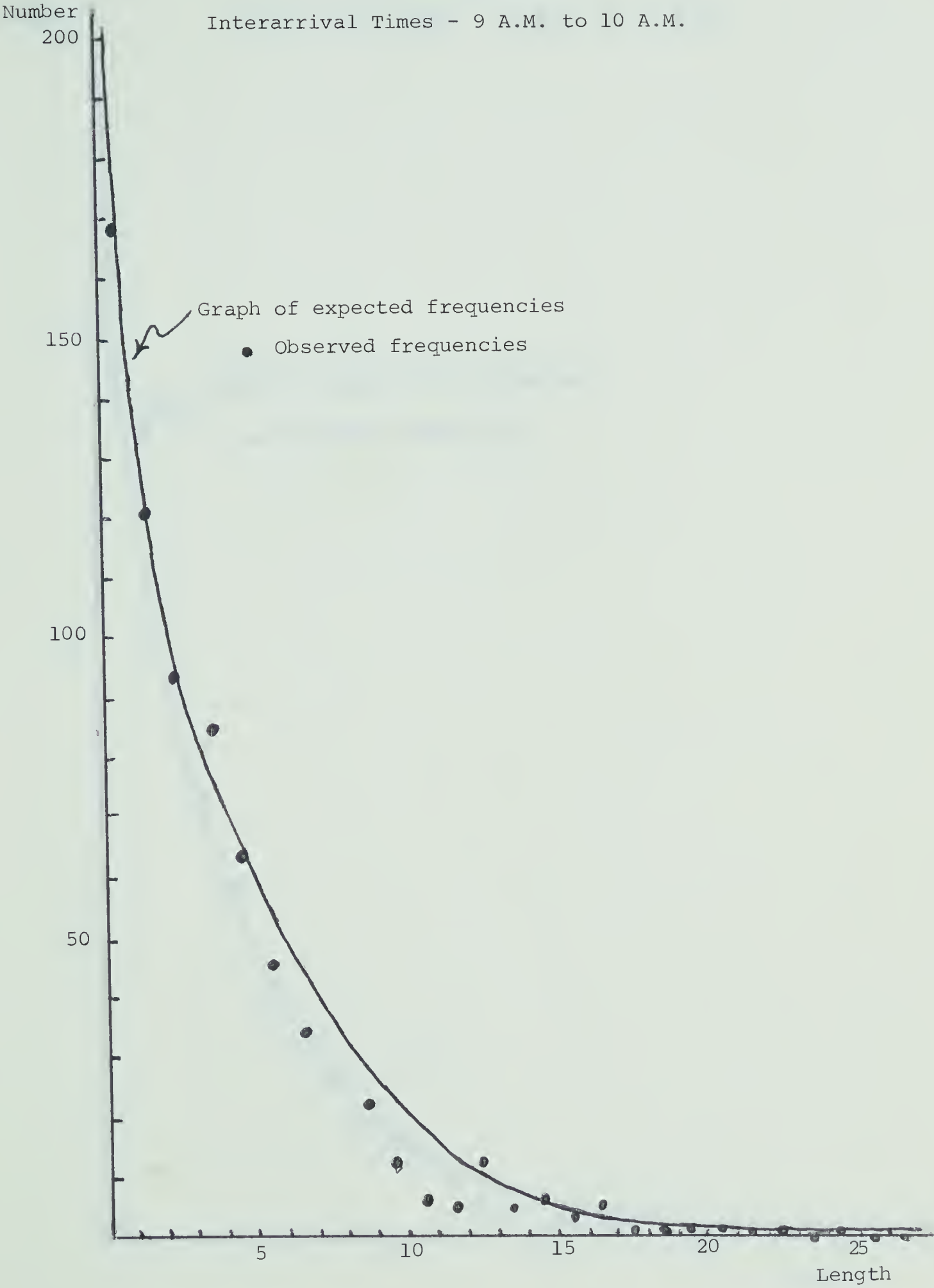




FIGURE 3.2

Interarrival Times - 10 A.M. to 11 A.M.

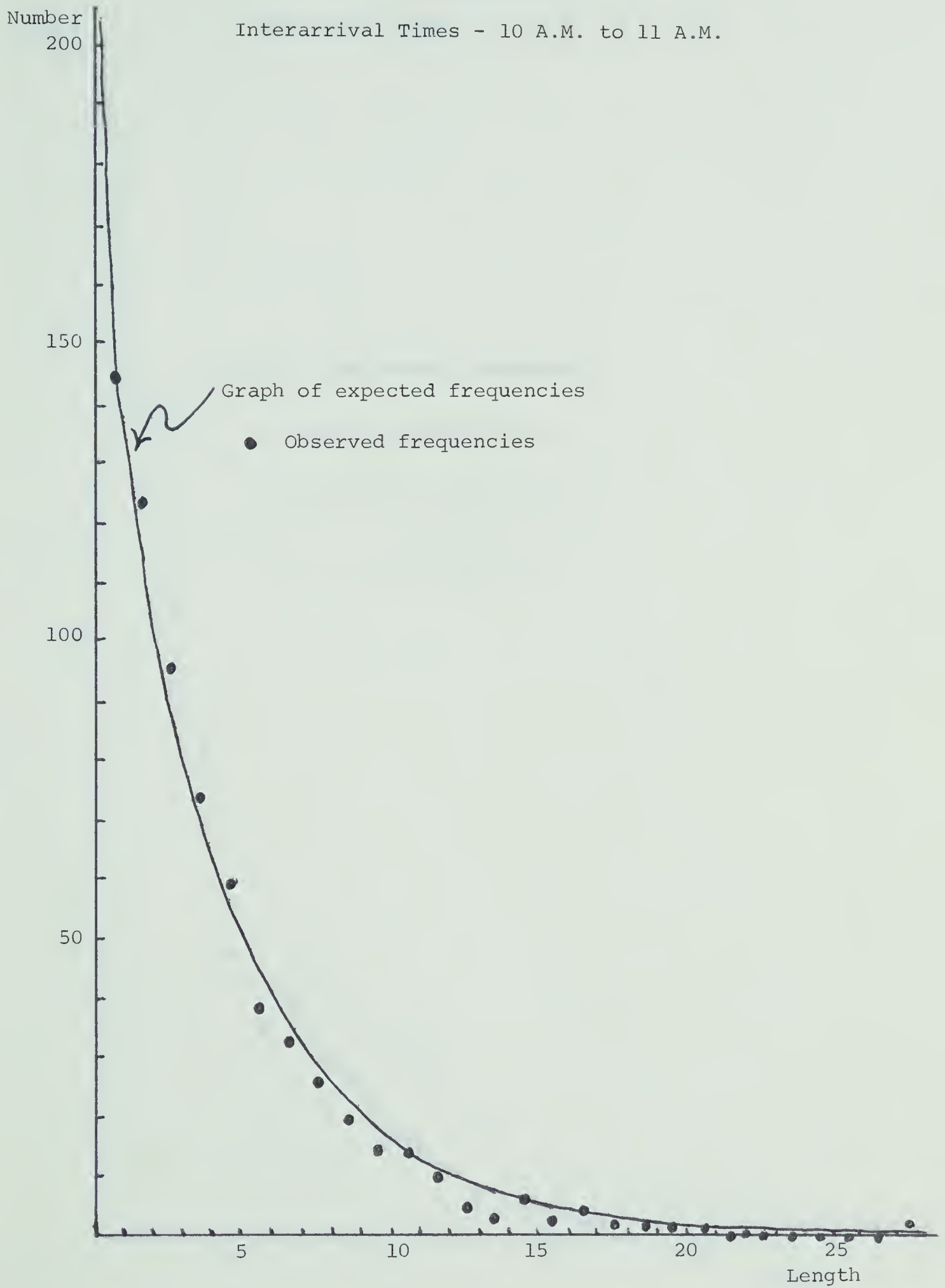




FIGURE 3.3

Interarrival Times - 11 A.M. to 12 P.M.

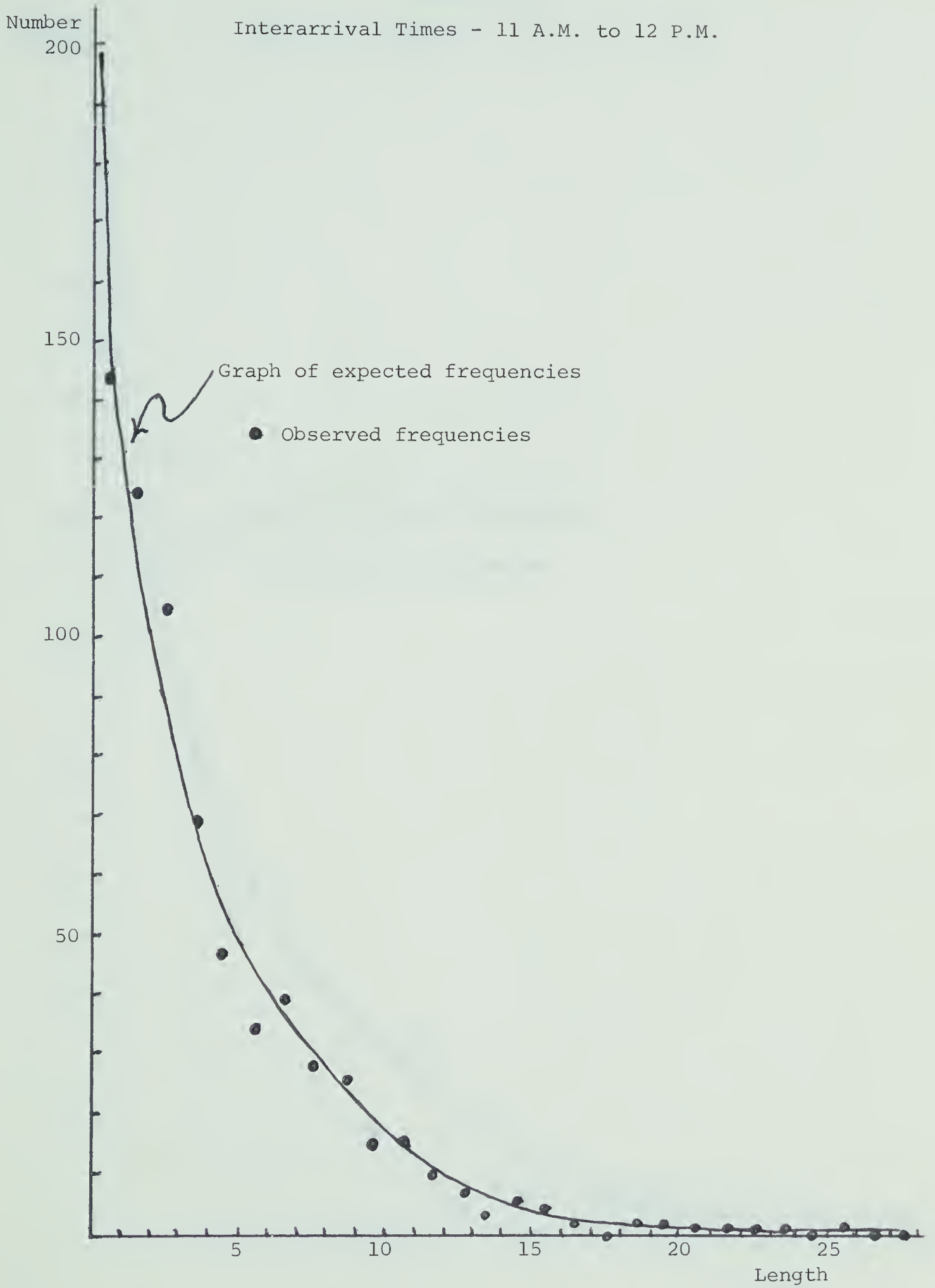




FIGURE 3.4

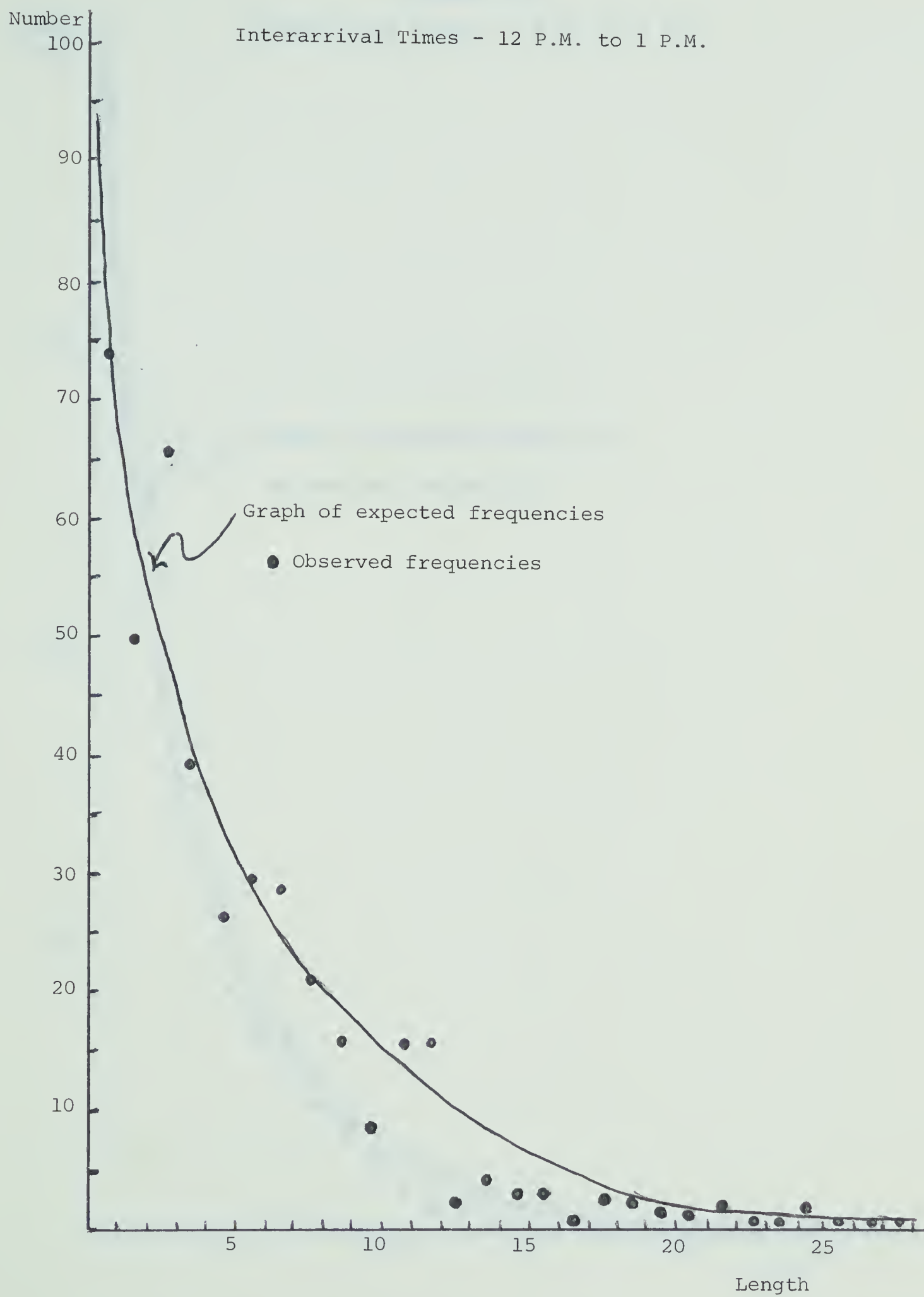




FIGURE 3.5

Interarrival Times - 1 P.M. to 2 P.M.

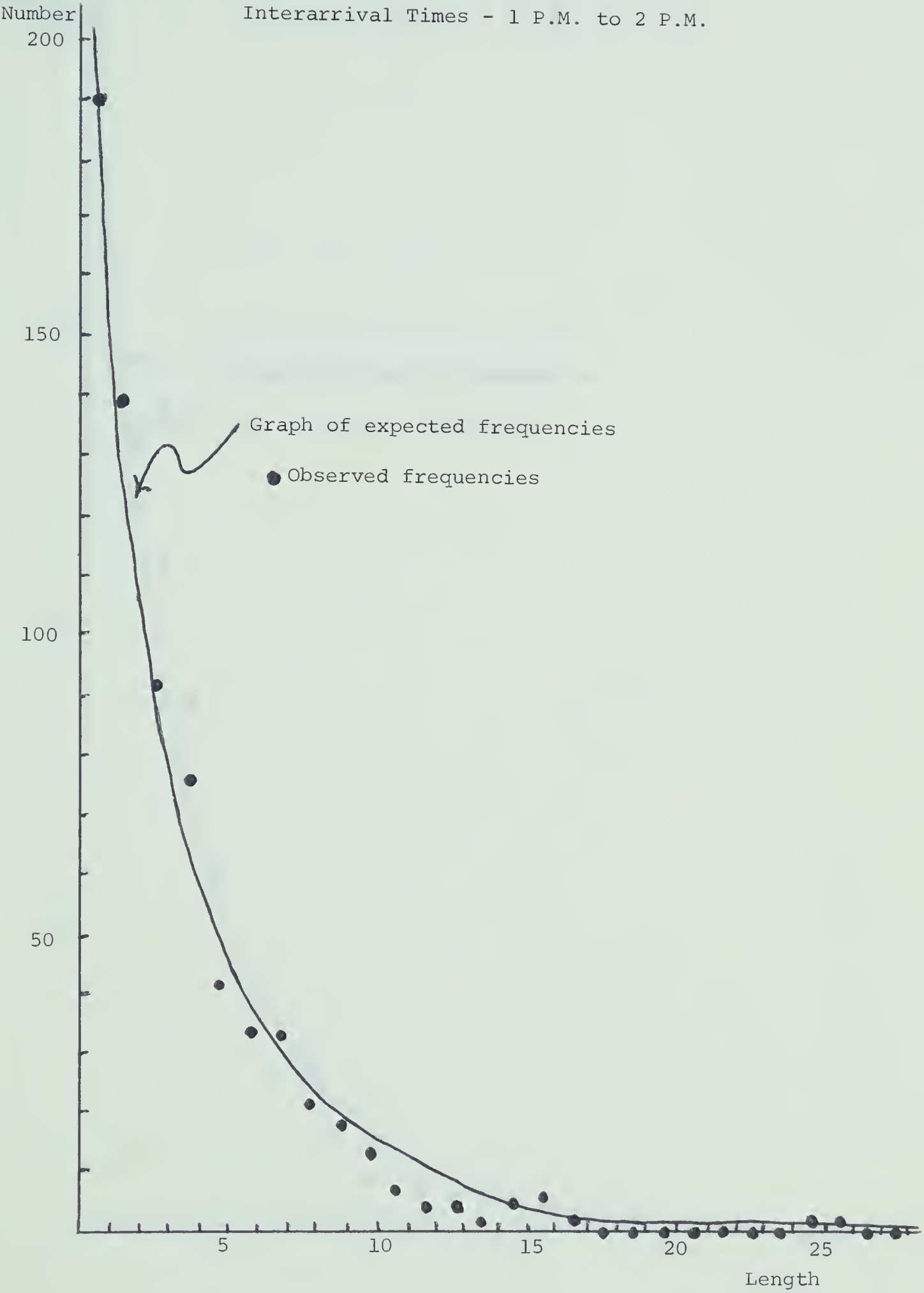




FIGURE 3.6
Interarrival Times - 2 P.M. to 3 P.M.

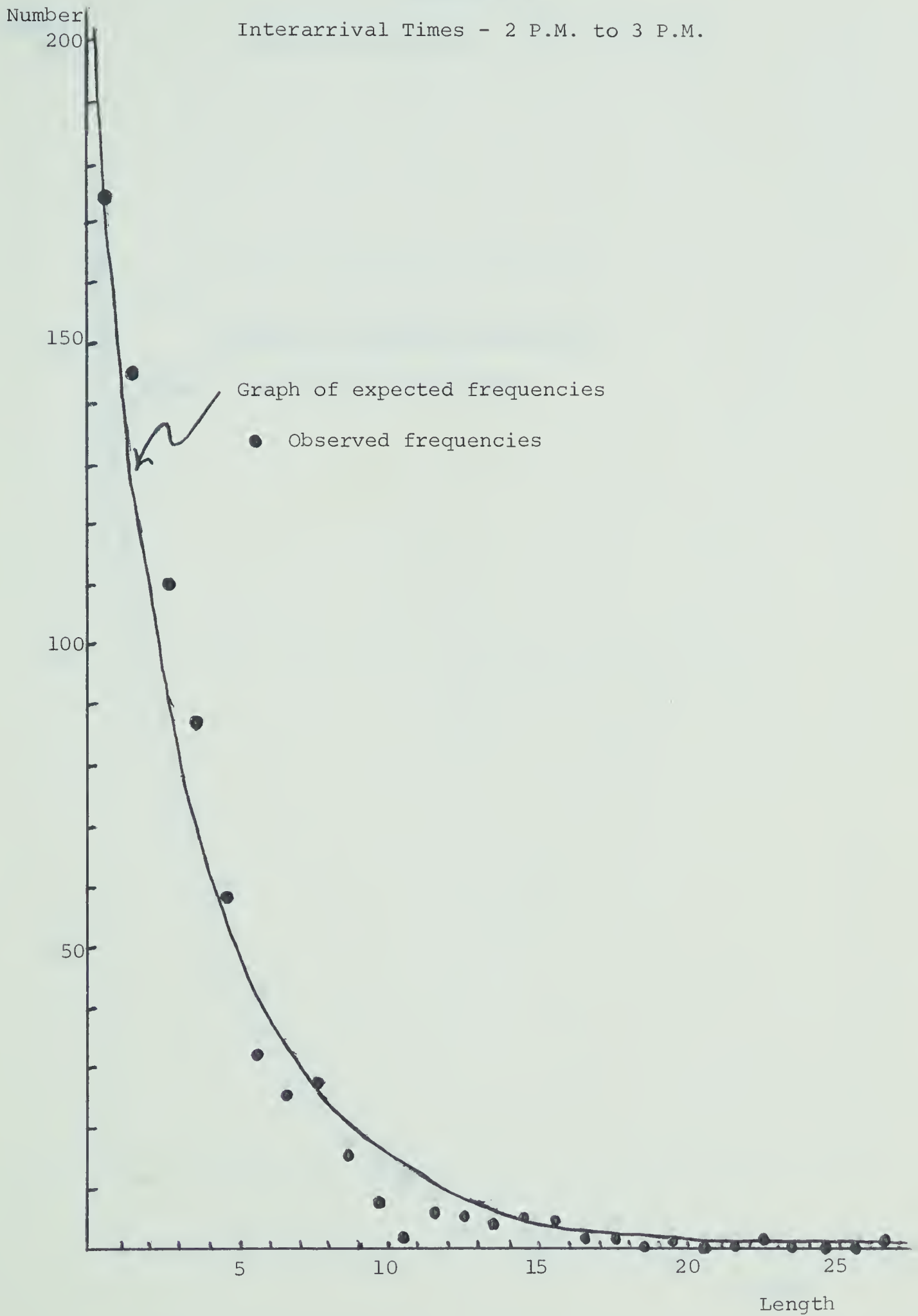




FIGURE 3.7

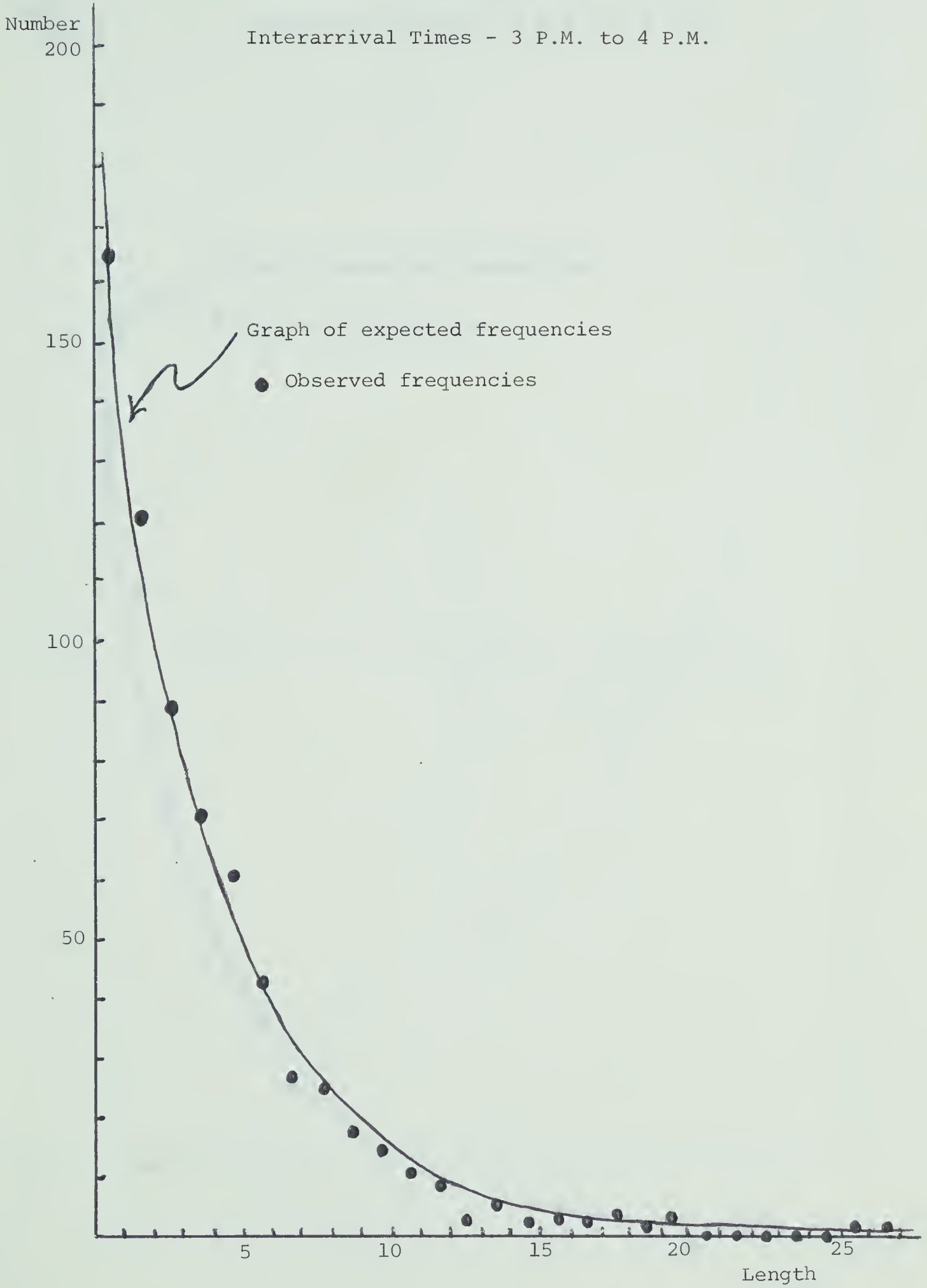
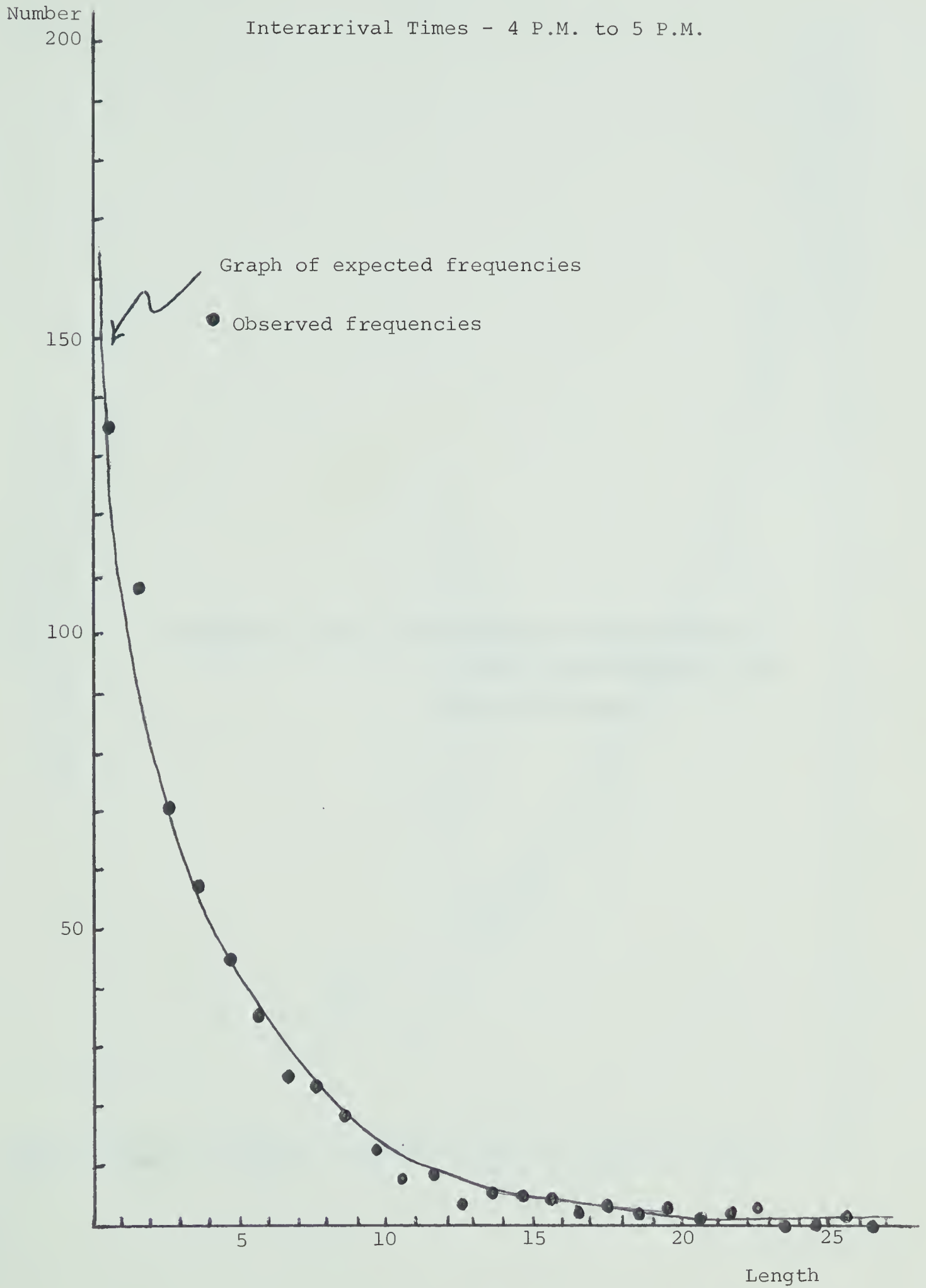




FIGURE 3.8

Interarrival Times - 4 P.M. to 5 P.M.





FIGURES 3.9 to 3.16: Graphical representations
of the busy times for each
one hour period

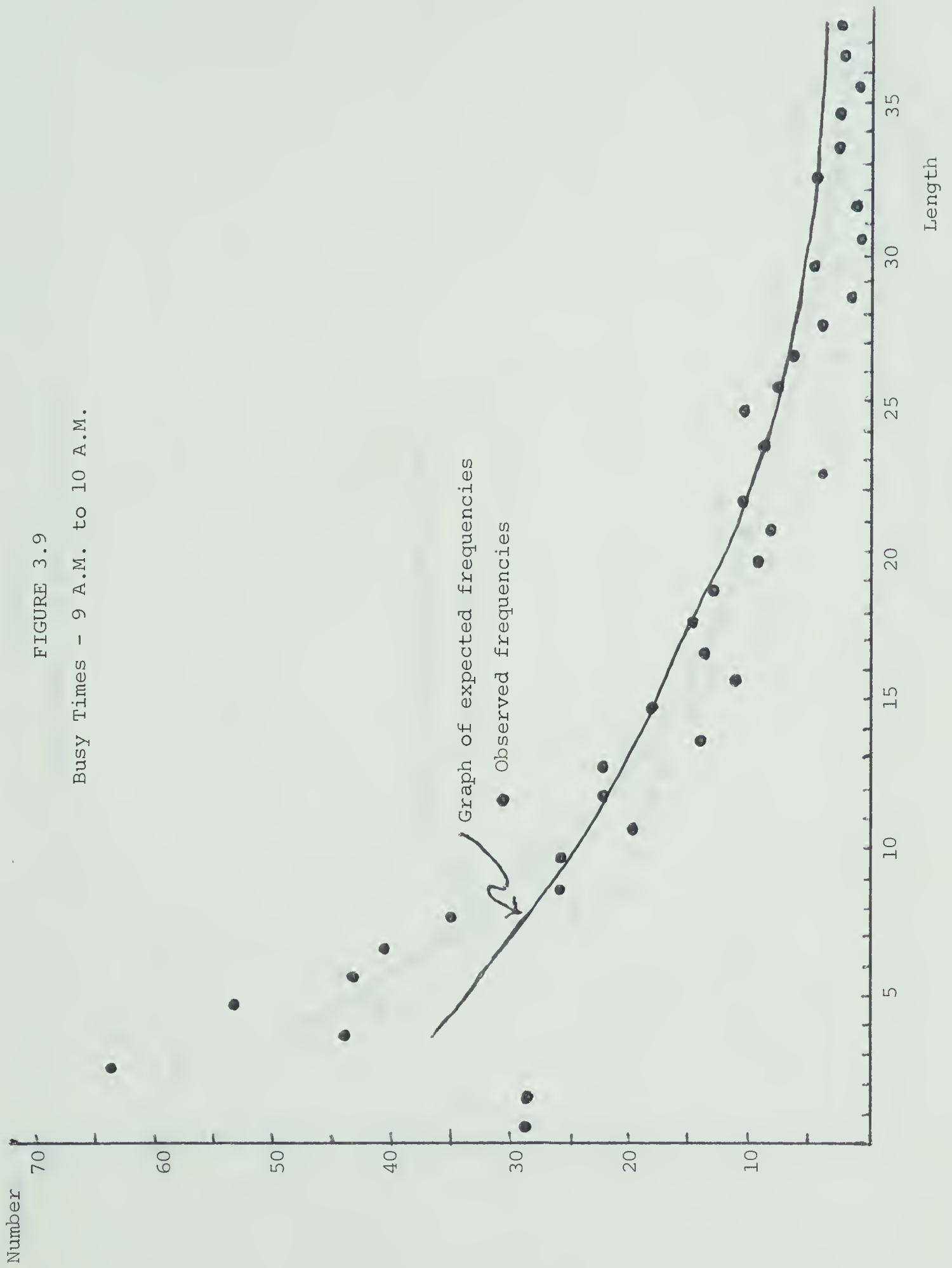
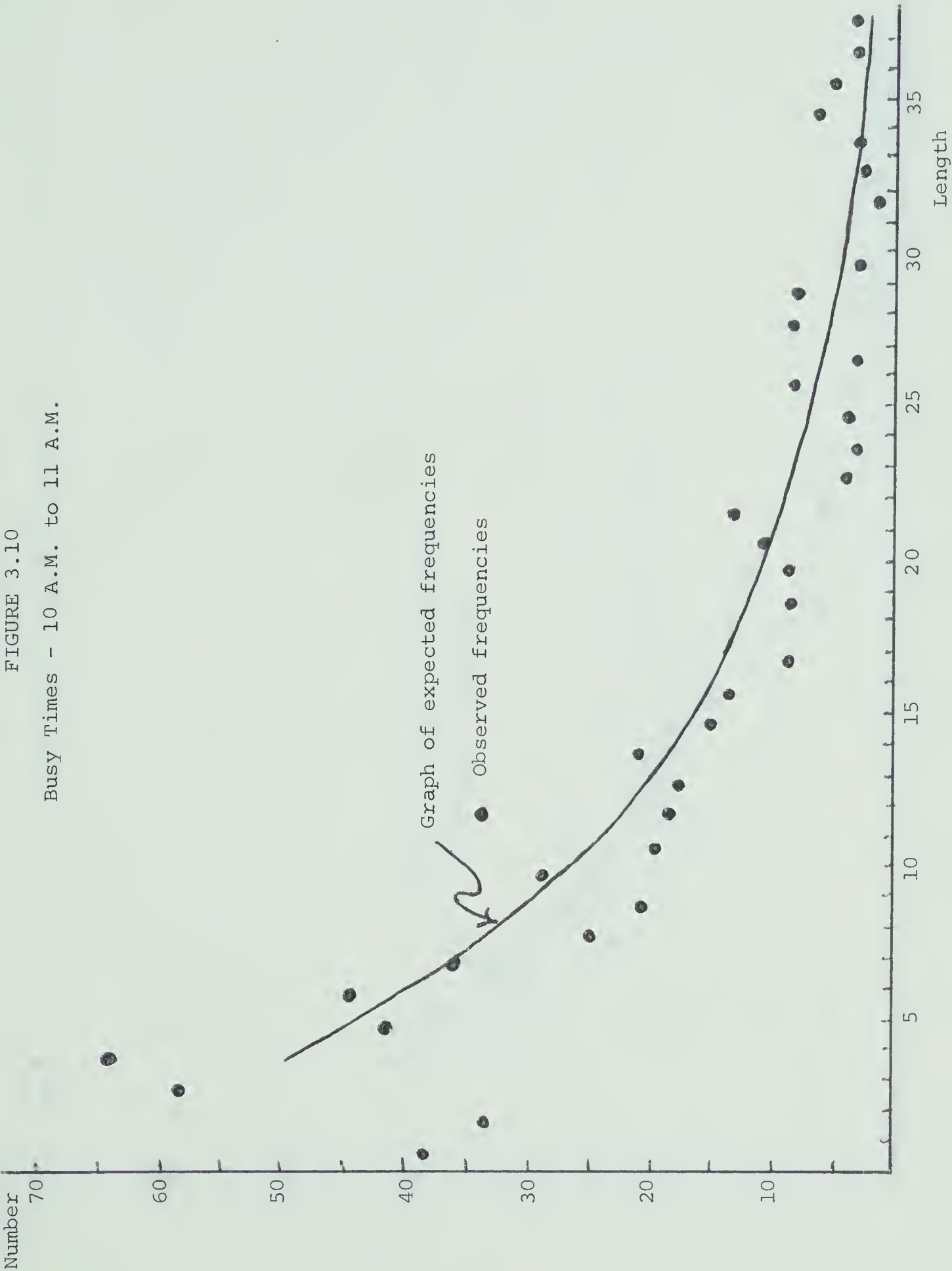


FIGURE 3.9

Busy Times - 9 A.M. to 10 A.M.

FIGURE 3.10
Busy Times - 10 A.M. to 11 A.M.



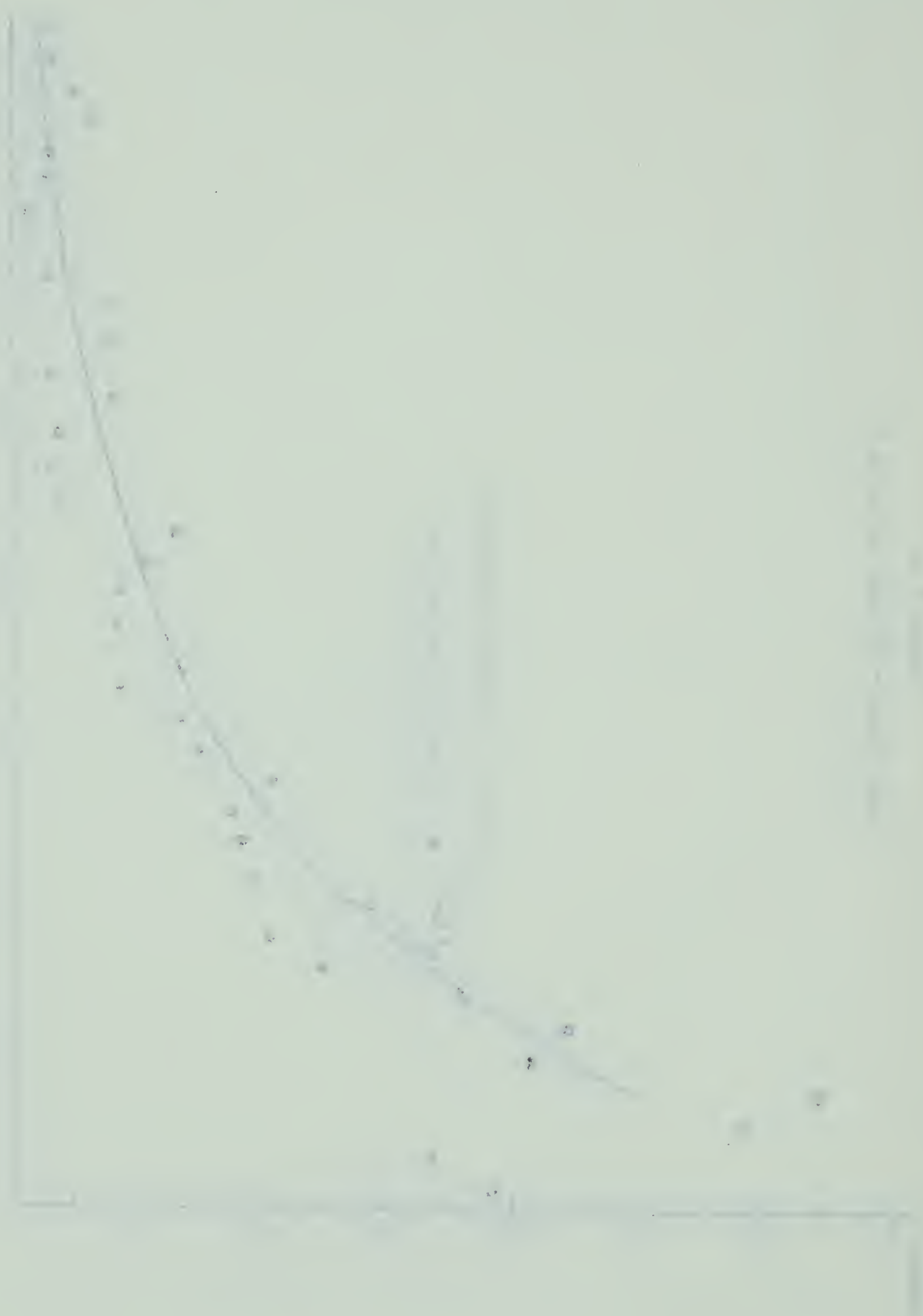


FIGURE 3.11
Busy Times - 11 A.M. to 12 P.M.

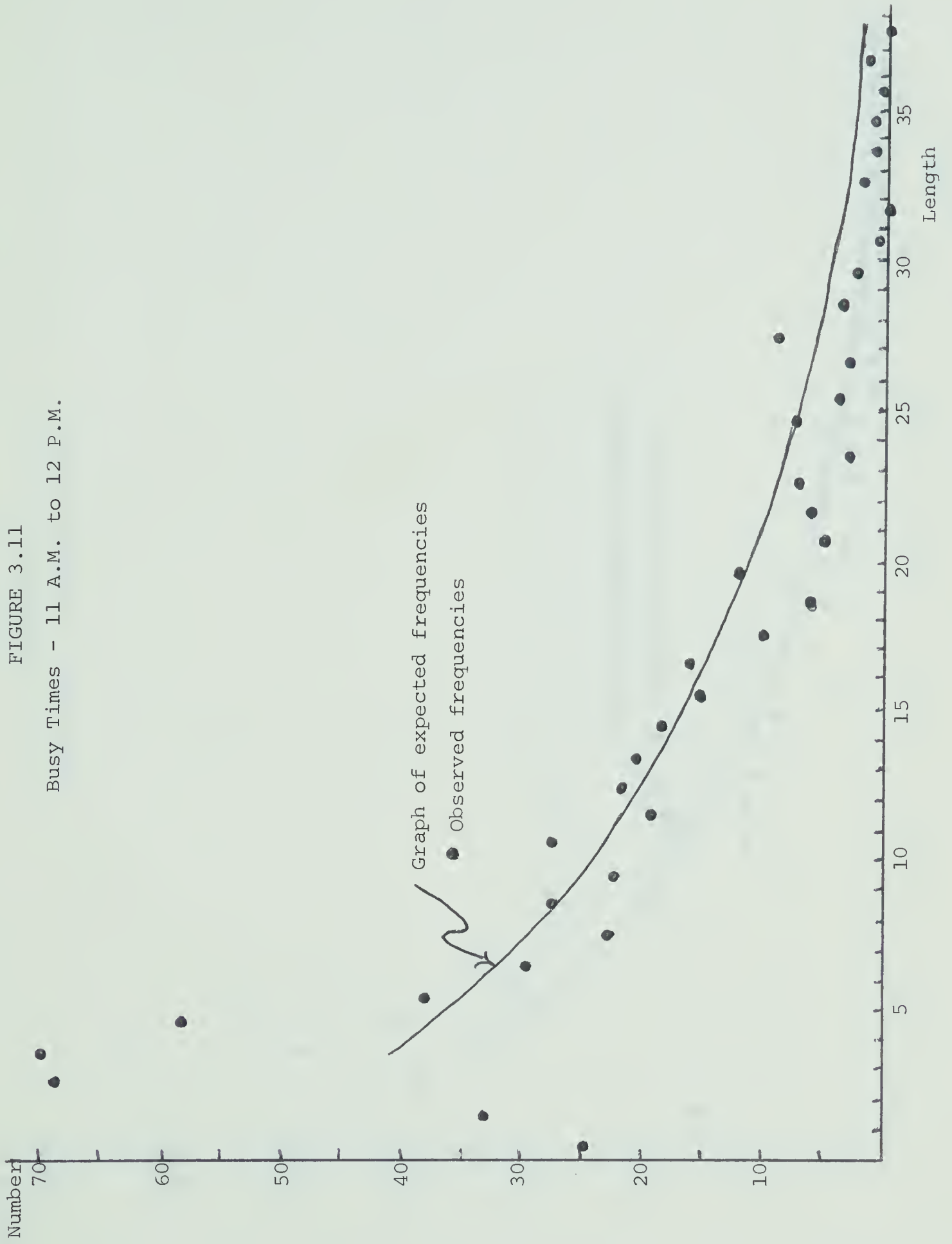


FIGURE 3.12
 Busy Times - 12 P.M. to 1 P.M.

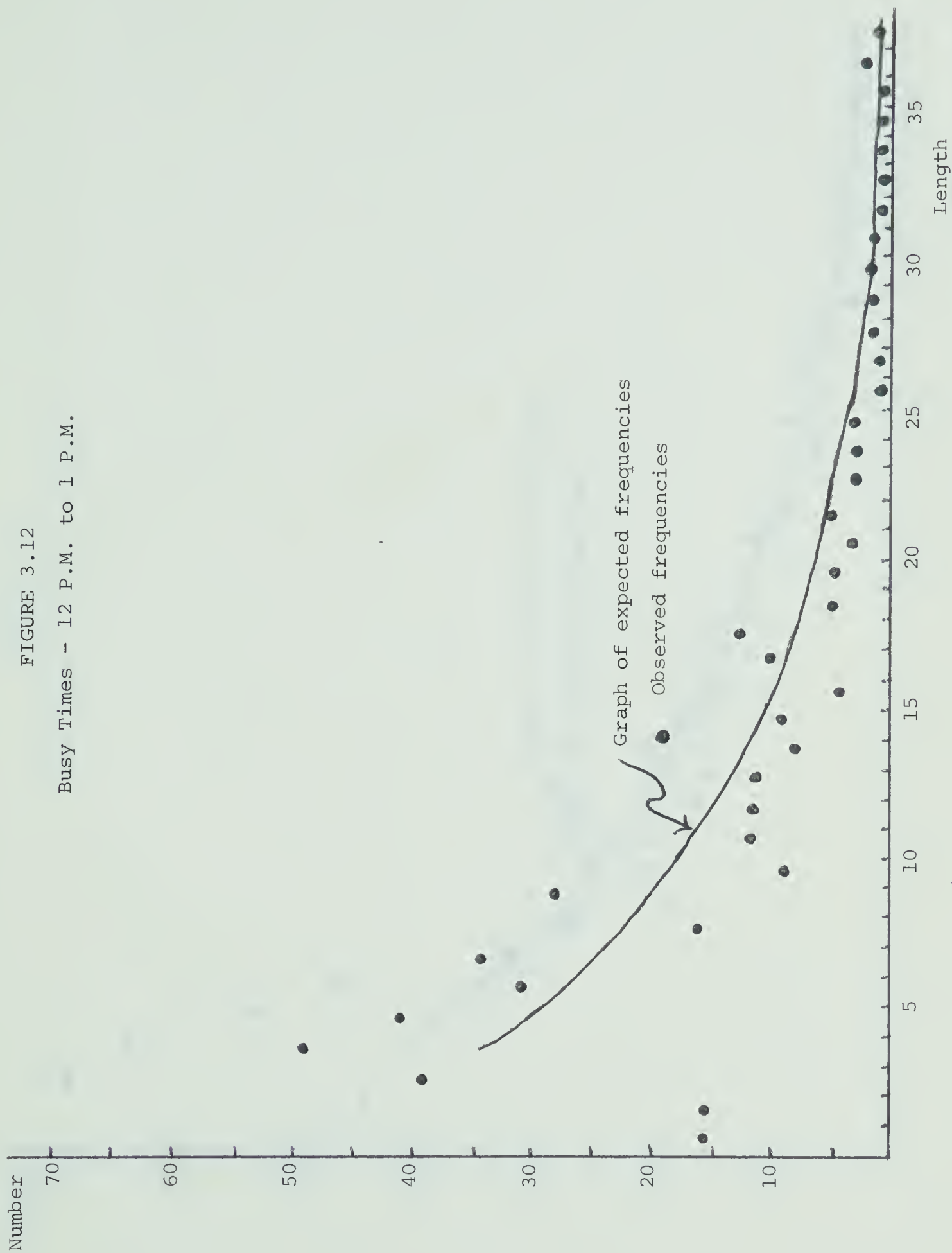
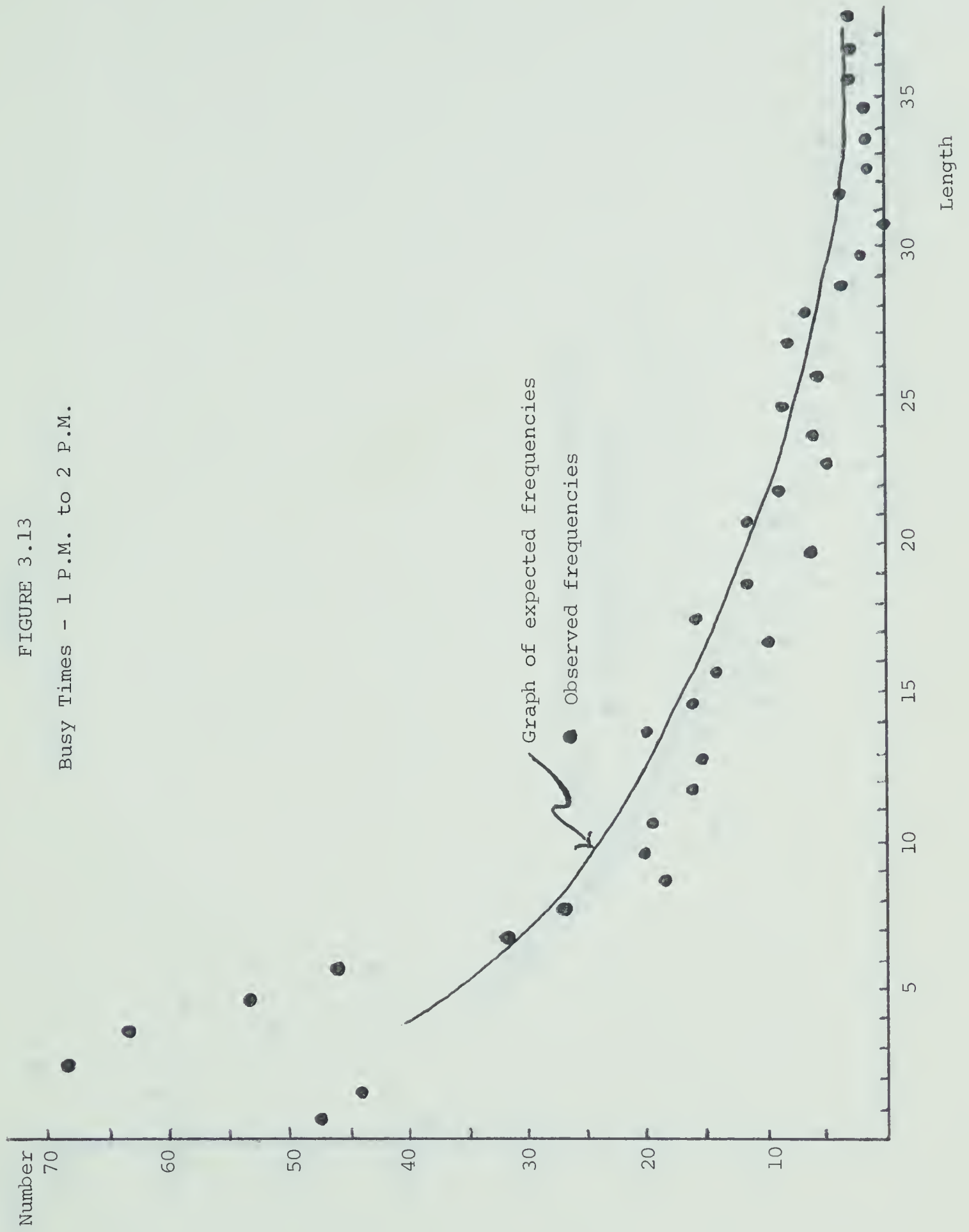


FIGURE 3.13
Busy Times - 1 P.M. to 2 P.M.





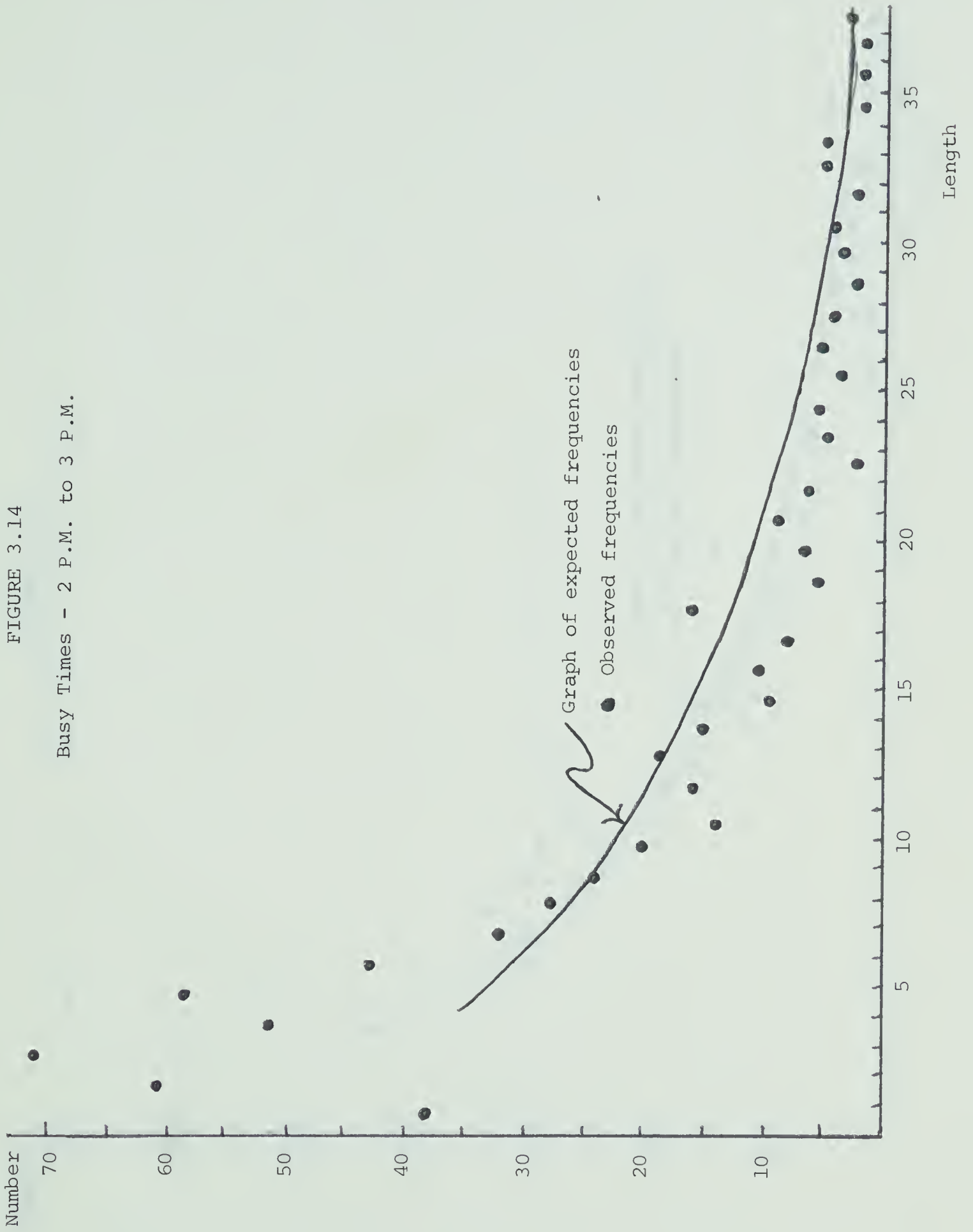


FIGURE 3.14

Busy Times - 2 P.M. to 3 P.M.



FIGURE 3.15
Busy Times - 3 P.M. to 4 P.M.

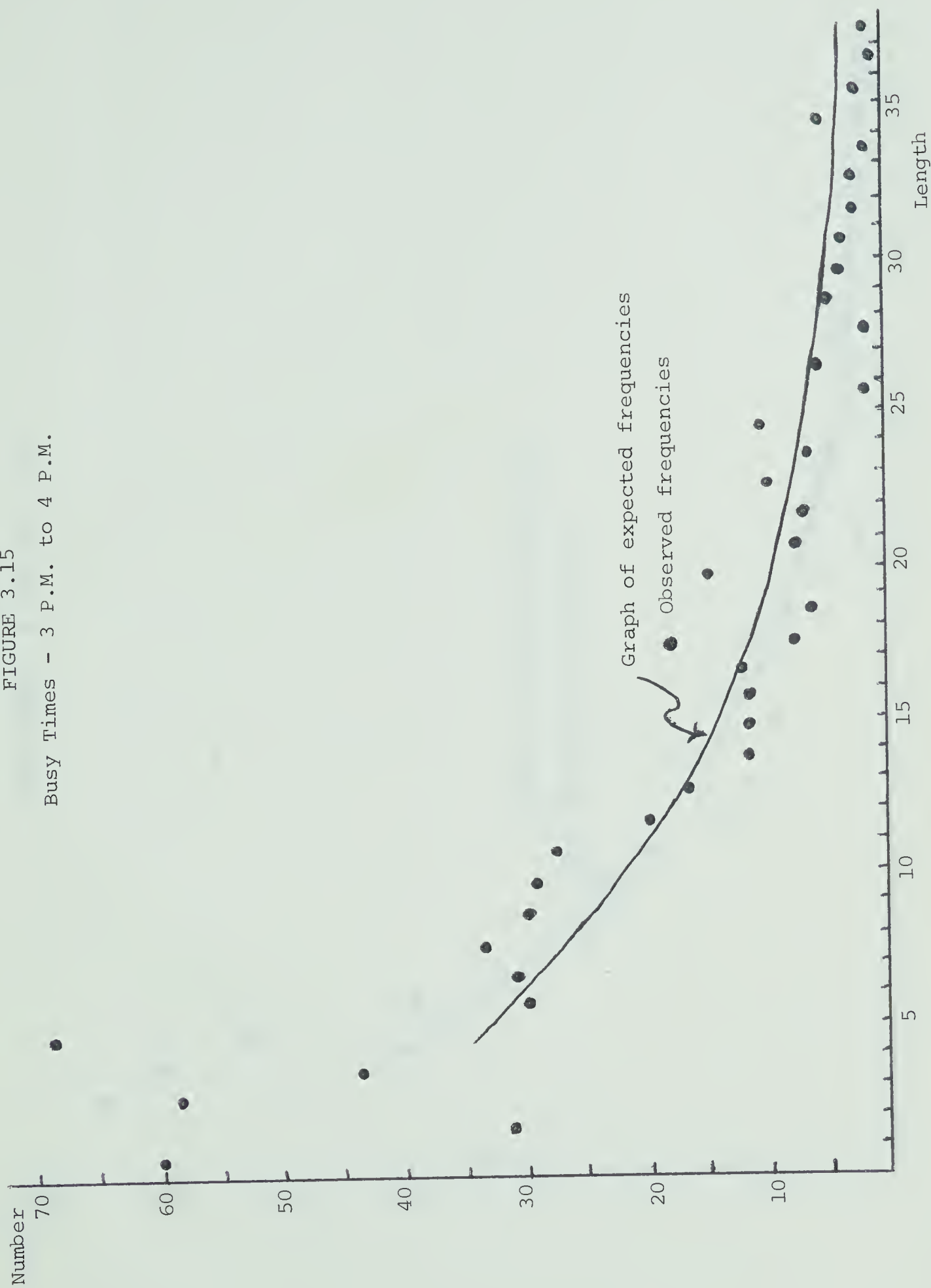
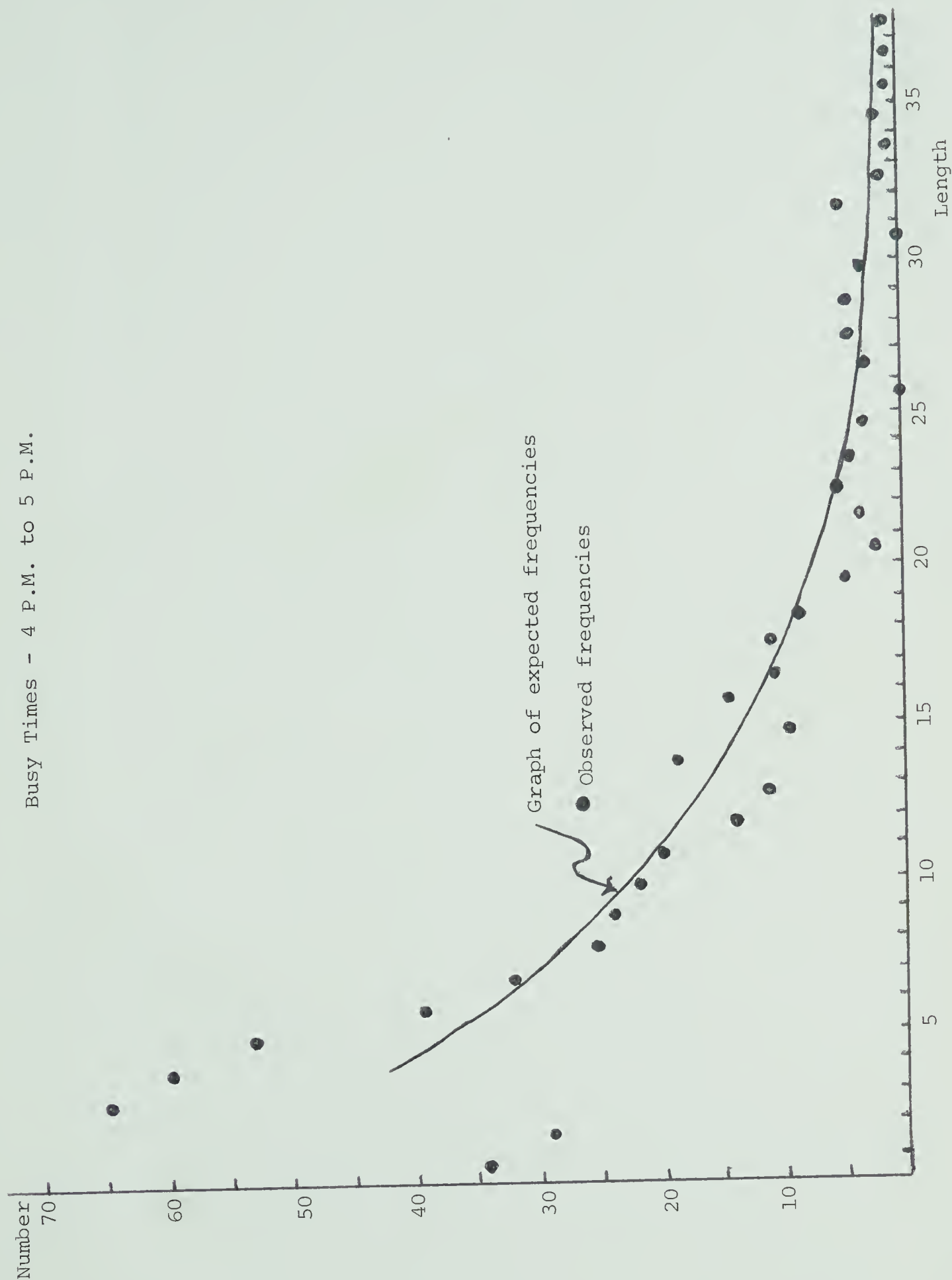
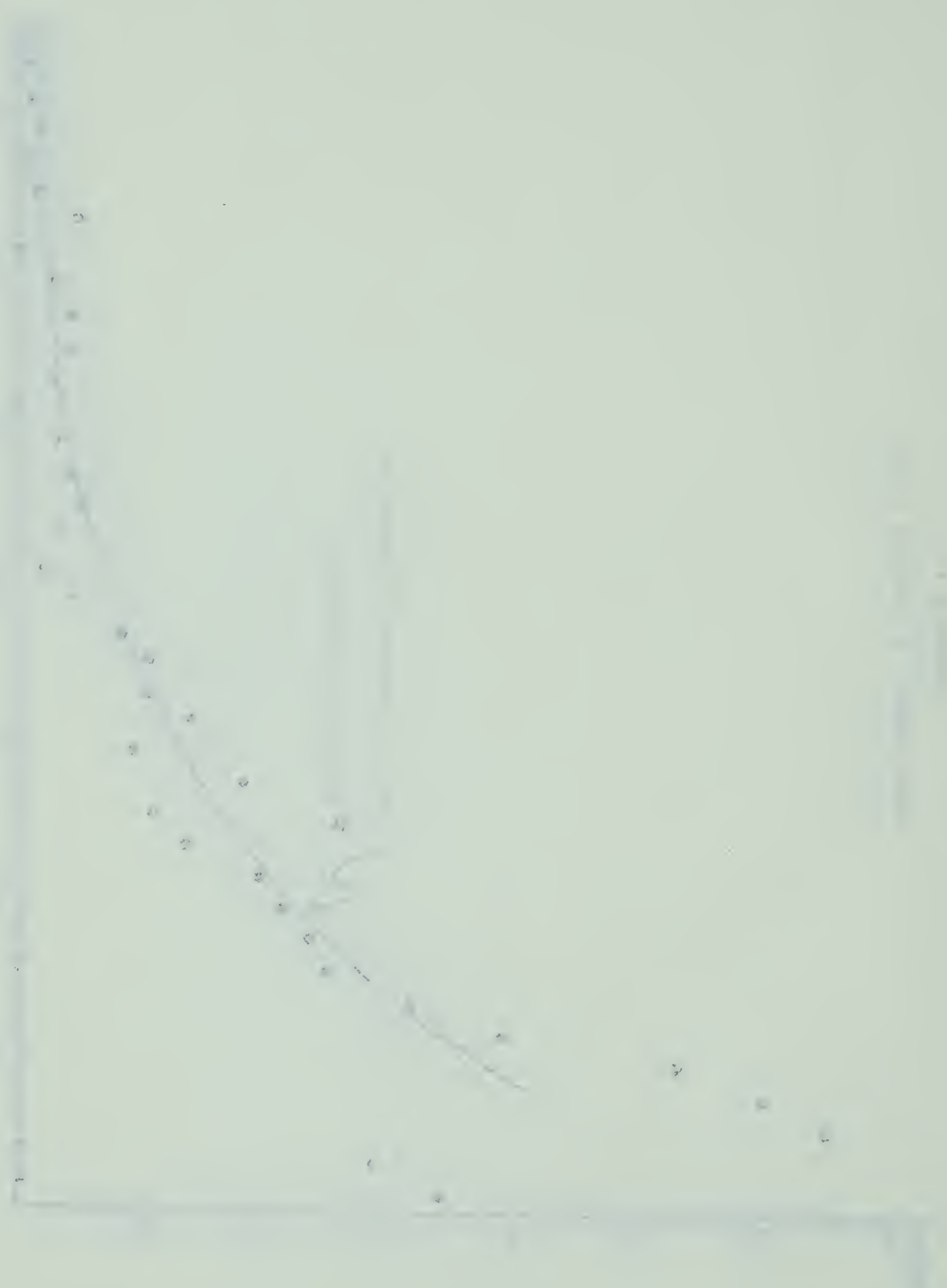




FIGURE 3.16
 Busy Times - 4 P.M. to 5 P.M.





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